Time-Variant Reliability Analysis by FORM and SORM

Armen Der Kiureghian
University of California, Berkeley

Joint Committee on Structural Safety
Technical University of Munich
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Outline

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FORM/SORM vs simulation and UQ methods

- FORM/SORM entail approximations that are affected by the degree of nonlinearity in the limit-state surface and the dimensionality of the problem.
- At the cost of large computational effort, simulation methods can estimate desired probabilities with specified accuracy.
- Accuracy of UQ methods depends on the accuracy of meta models and methods used for computing the failure probability.
- Simulation and UQ methods are primarily aimed at computing the failure probability.
- FORM/SORM provide additional valuable information:
  - Most likely realization of random variables causing failure
  - Relative importance of random variables
  - Sensitivities with respect to parameters in probability distributions and the limit-state model
  - Physical interpretation of the linearized system

Notation

Notation:

t time

T time interval

$\mathbf{X}(t)$ vector of random processes

$g(\mathbf{X}(t), t)$ limit-state function

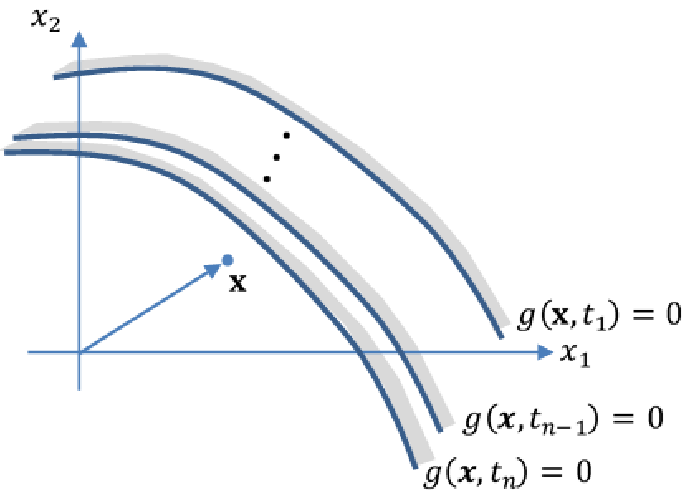
$\mathcal{F} = \left\{ \min_{t \in T} g[\mathbf{X}(t), t] \leq 0 \right\}$ failure domain

Types of problems

Encroaching problem:

$g(\mathbf{X}, t)$

Failure occurs when the limit-state surface "encroaches" onto the outcome point $\mathbf{X} = \mathbf{x}$.



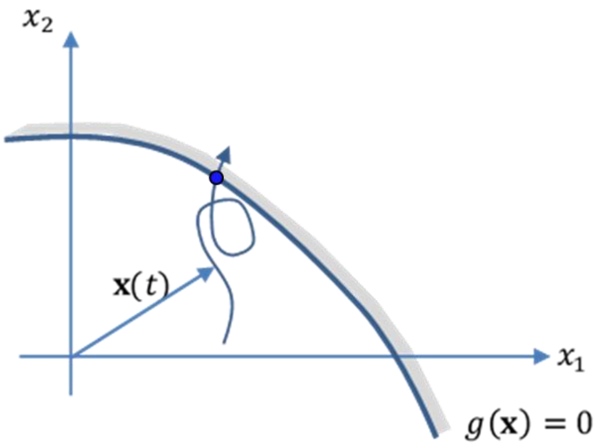
- Example: a structure with a capacity that deteriorates according to a deterministic rule.

Types of problems

Outcrossing problem:

$g[\mathbf{X}(t)]$

Failure occurs when the random vector process $\mathbf{X}(t)$ out-crosses the limit-state surface.



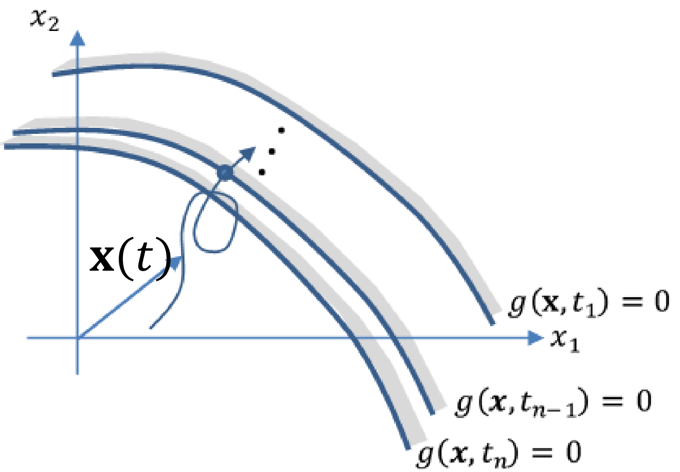
- Example: a structure under stochastic loads.

Types of problems

Encroaching-outcrossing problem:

$g[\mathbf{X}(t), t]$

Failure occurs when the random vector process $\mathbf{X}(t)$ out-crosses the encroaching limit-state surface.



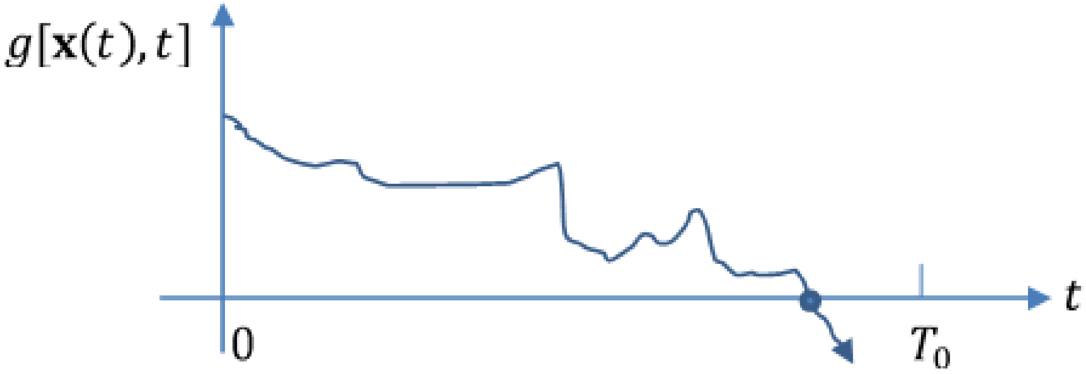
- Example: a structure under stochastic loads and a capacity that deteriorates according to a deterministic rule.

Types of problems

Alternative way of formulating the problem:

$g[\mathbf{X}(t), t]$ is a scalar quantity for give t.

Failure occurs when $g[\mathbf{X}(t), t]$ down-crosses the zero level during $t \in T$.



Solutions for time-variant reliability

Upper-bound solution:

$N(T)$ = Number of zero-level down-crossings of $g[\mathbf{X}(t),t]$ in $t \in T$

$$\Pr\left\{\min_{t\in T} g[\mathbf{X}(t),t] \le 0\right\} = \Pr\{g[\mathbf{X}(0),0] \le 0 \cup 0 < N(T)\}$$
$$\le \Pr\{g[\mathbf{X}(0),0] \le 0\} + \Pr[0 < N(T)]$$

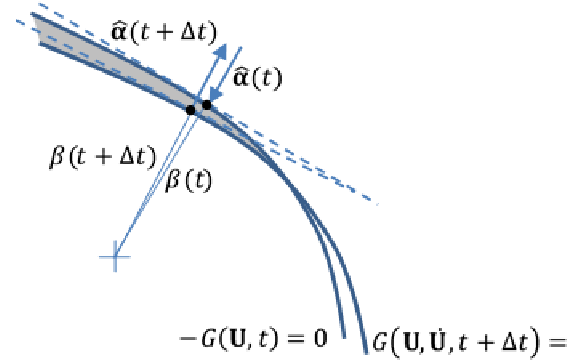
$$\Pr[0 < N(T)] \le \int_T \nu(t)dt$$

$\nu(t)$ = mean rate of zero-level down-crossings of $g[\mathbf{X}(t),t]$

- $\Pr\left\{\min_{t\in T} g[\mathbf{X}(t),t] \le 0\right\} \le \Pr\{g[\mathbf{X}(0),0] \le 0\} + \int_T \nu(t)dt$

Solutions for time-variant reliability

Upper-bound FORM solution:



$$\nu(t) = \lim_{\Delta t\to 0}\frac{\Pr\{-g[\mathbf{X}(t),t]\le 0 \cap g[\mathbf{X}(t+\Delta t),t+\Delta t]\le 0\}}{\Delta t}$$

$$\cong \frac{\Pr\{-g[\mathbf{X}(t),t]\le 0 \cap g\left[\mathbf{X}(t)+\dot{\mathbf{X}}(t)\Delta t,t+\Delta t\right]\le 0\}}{\Delta t}$$

$$\cong \frac{\Phi_2[-\beta(t),-\beta(t+\Delta t),\rho(t,\Delta t)]}{\Delta t}$$

$\beta(t)$ = reliability index of problem $\{-g[\mathbf{X}(t),t]\le 0\}$

$\beta(t+\Delta t)$ = reliability index of problem $\{g\left[\mathbf{X}(t)+\dot{\mathbf{X}}(t)\Delta t,t+\Delta t\right]\le 0\}$

$\rho(t,\Delta t) = \hat{\boldsymbol{\alpha}}(t)\hat{\boldsymbol{\alpha}}(t+\Delta t)^{\mathrm{T}}$

For small Δt, $\beta(t)\cong-\beta(t+\Delta t)$ and $-1\le\rho(t,\Delta t)$:

- $\nu(t)\cong\frac{1}{\Delta t}\exp\left[\frac{\beta(t)\beta(t+\Delta t)}{2}\right]\left\{\frac{1}{4}+\frac{\sin^{-1}[\rho(t,\Delta t)]}{2\pi}\right\}$

Solutions for time-variant reliability

Poisson approximation:

Assume zero-level down-crossings are Poisson events (reasonable when these events are rare):

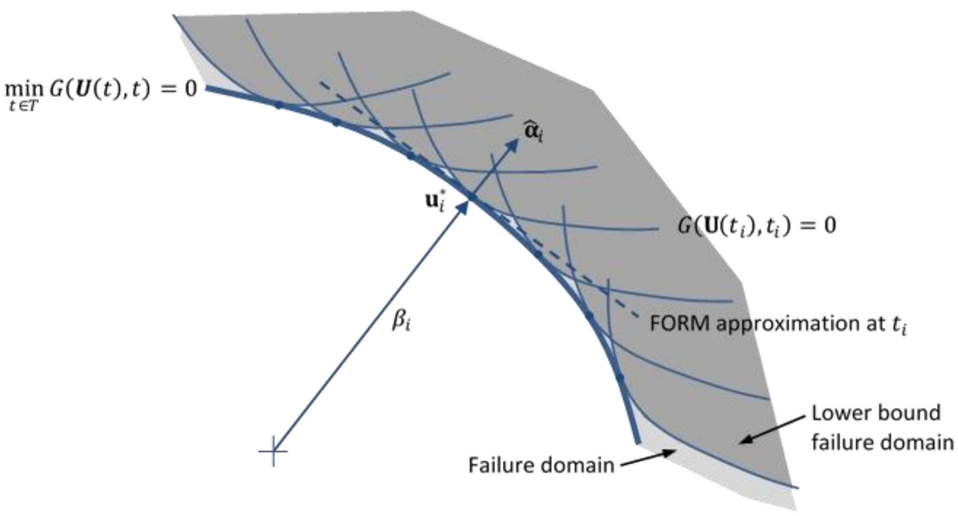
- $\Pr\left\{\min_{t \in T} g[\mathbf{X}(t), t] \le 0\right\} \cong \Pr\{g[\mathbf{X}(0), 0] \le 0\} + 1 - \exp\left[-\int_T \nu(t)\mathrm{d}t\right]$
- $f_{T_1}(t) = \nu(t)\exp\left[-\int_0^t \nu(t)\mathrm{d}t\right], \quad 0 < t$ PDF of time to failure

Solutions for time-variant reliability

Lower-bound solution as a series system:

$$\Pr\left\{\min_{t\in T} g[\mathbf{X}(t),t] \le 0\right\} \ge \Pr\{\cup_{i=1}^{n}\{g[\mathbf{X}(t_i),t_i] \le 0\}\}$$

$$\cong 1-\Phi_n(\mathbf{B},\mathbf{R}) \quad \text{FORM approximation}$$

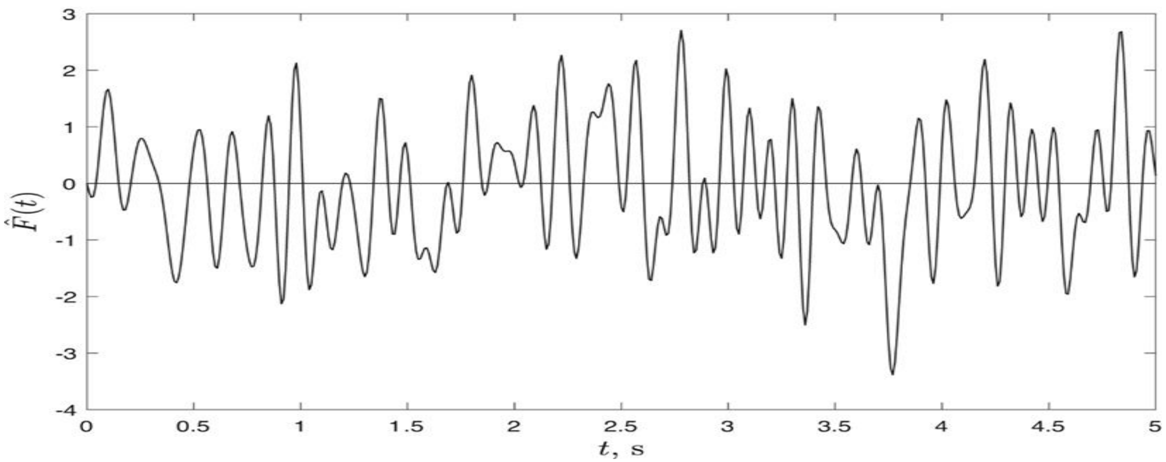


Requires solutions at a large number of time points.

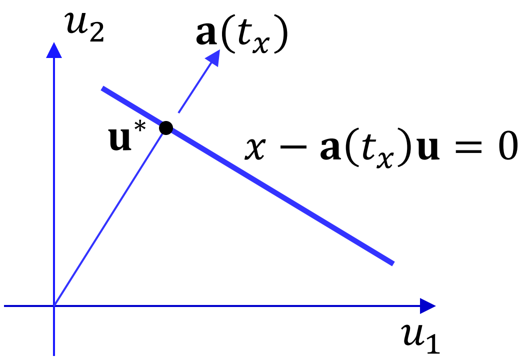
Stochastic dynamic problems

Discrete representation of input stochastic process:

- $\hat{F}(t) = \mu_F(t) + \mathbf{s}(t)\mathbf{U}, \quad \mathbf{U} = \mathrm{N}(\mathbf{0}, \mathbf{I})$
  - $s_i(t) = \sigma, \quad t_{i-1} < t \le t_i$ $\quad = \sigma \,\mathrm{sinc}_{\Delta t}(t - i\Delta t)$ discrete white noise representation
  - $s_i(t) = \sigma q(t) \int_{t_{i-1}}^{t_i} h_F(t-\tau)\mathrm{d}\tau$ modulated, filtered white noise



Response of linear system to Gaussian excitation



- $X(t) = \int_0^t \hat{F}(\tau)h(t-\tau)\mathrm{d}\tau$

  $= \mu_X(t) + \mathbf{a}(t)\mathbf{U}, \quad a_i(t) = \int_0^t s_i(\tau)h(t-\tau)\mathrm{d}\tau$

- $G(\mathbf{U}, x, t_x) = x - \mathbf{a}(t_x)\mathbf{U}$
  - $\mathbf{u}^*(x, t_x) = \frac{x\mathbf{a}(t_x)}{\|\mathbf{a}(t_x)\|^2}$ design point
  - $\hat{f}^*(t) = \mu_F(t) + x\frac{\mathbf{s}(t)\mathbf{a}(t_x)^{\mathrm{T}}}{\|\mathbf{a}(t_x)\|^2}$ design point excitation
  - $x^*(t) = \mu_X(t) + x\frac{\mathbf{a}(t)\mathbf{a}(t_x)^{\mathrm{T}}}{\|\mathbf{a}(t_x)\|^2}$ design point response
  - $\hat{\boldsymbol{\alpha}}(x, t_x) = \frac{\mathbf{a}(t_x)}{\|\mathbf{a}(t_x)\|}$ unit normal vector,
  - $\beta(x, t_x) = \frac{x}{\|\mathbf{a}(t_x)\|}$ reliability index
  - $\Pr\{\mu_X(t_x) + x \le X(t_x)\} = \Phi[-\beta(x, t_x)]$ tail probability
  - $\boxed{\mathbf{a}(t_x) = \frac{x\mathbf{u}^*(x,t_x)^{\mathrm{T}}}{\|\mathbf{u}^*(x,t_x)\|^2}}$ cornerstone relation for TELM

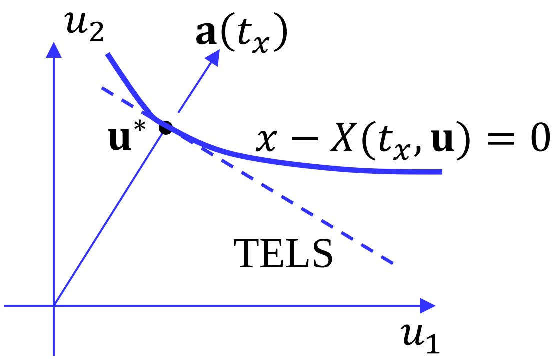
Response of nonlinear system to Gaussian excitation

- $G(\mathbf{U}, x, t_x) = x - X(t_x, \mathbf{U})$

  Determine the design point $\mathbf{u}^*$ by FORM analysis and compute the normal vector

  $$\mathbf{a}(t_x) = \frac{x\mathbf{u}^*(x, t_x)^{\mathrm{T}}}{\|\mathbf{u}^*(x, t_x)\|^2}$$

  This uniquely defines the **Tail-Equivalent Linear System (TELS)**.



- $\mathbf{a}(t_x)$ uniquely defines the unit Impulse Response Function of the linear system. The design point excitation closely resembles the mirror image of the IRF.

- Tail probability of TELS equals FORM approximation of the tail probability of the nonlinear system – **Tail-Equivalent Linearization Method (TELM)**.

Properties of TELS/TELM

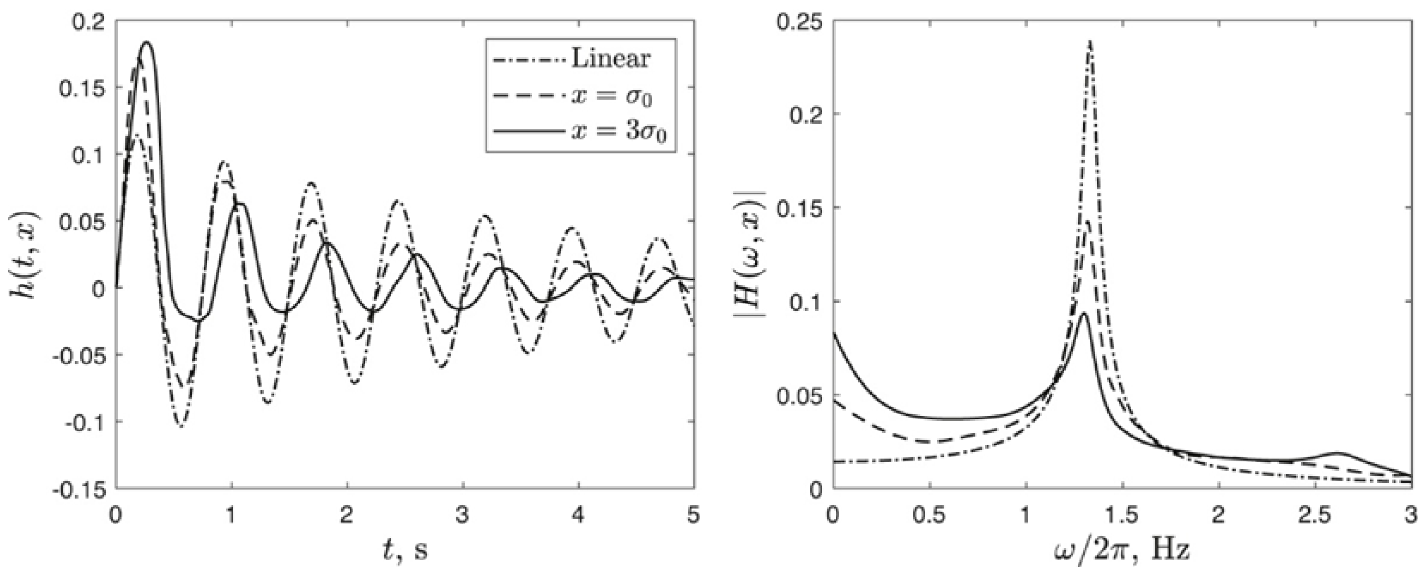
- Tail probability of TELS equals FORM approximation of the tail probability of the nonlinear system.
- TELM is a non-parametric linearization method – no parametrized linear model is needed; TELS is determined numerically in terms of its IRF.
- TELS critically depends on the assumed threshold x, thus capturing the non-Gaussian distribution of the nonlinear response.
- For stationary response, analysis at a single time point is sufficient.
- TELS is independent of the scaling of the input excitation. For excitation $cF(t)$, the reliability index is $\beta(x, t_x)/c$, where $\beta(x, t_x)$ is the reliability index for $F(t)$. This property greatly simplifies fragility analysis.
- For the TELS to exist, the loading history of the nonlinear system must be differentiable. (Bilinear elasto-plastic model will not work. Must work with smoothened models.)

Example – Hysteretic oscillator subjected to band-limited white noise base excitation

- $m\ddot{X}(t) + c\dot{X}(t) + k[\alpha X(t) + (1-\alpha)Z(t) = -mF(t)$

  $\dot{Z}(t) = -\gamma\left|\dot{X}(t)\right||Z(t)|^{n-1}Z(t) - \eta|Z(t)|^{n}\dot{X}(t) + A\dot{X}(t),$

  $m = 300{,}000\text{kg}$, $c = 150\text{kNs/m}$, $k = 21{,}000\text{kN/m}$, $\alpha = 0.1$, $n = 3$, $A = 1$,
  $\gamma = \eta = 1(2\sigma_0^n),\ \sigma_0^2 = \pi S m^2/(ck)$ for $\alpha = 1$ case.



IRF (left) and FRF (right) of the TELS for different thresholds

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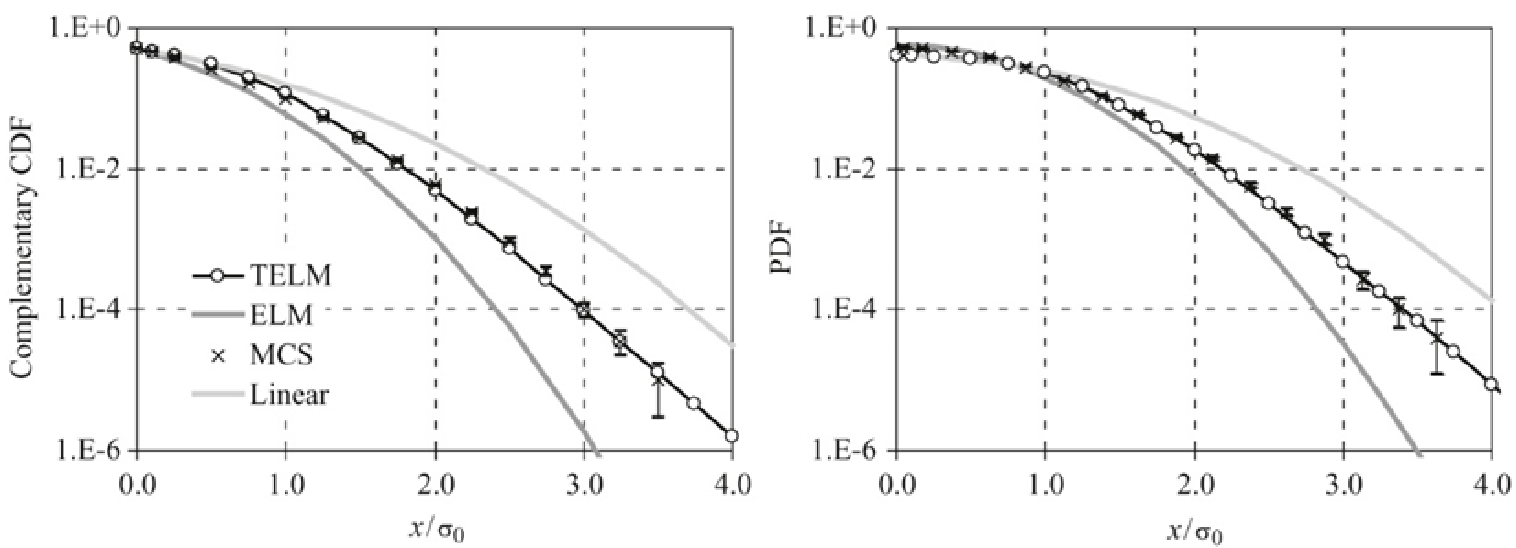

Point-in-time complementary CDF (left) and PDF (right) of hysteretic response

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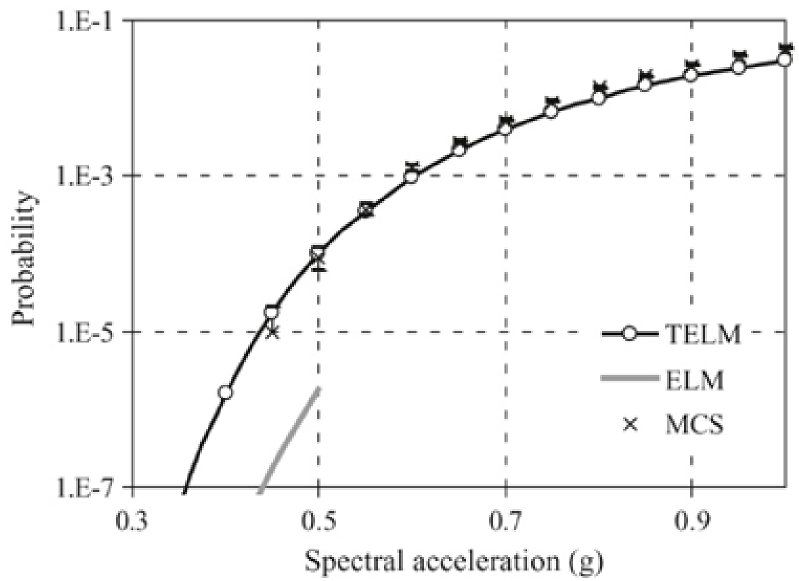

Fragility curve for tail probability at threshold $x = 3\sigma_0$

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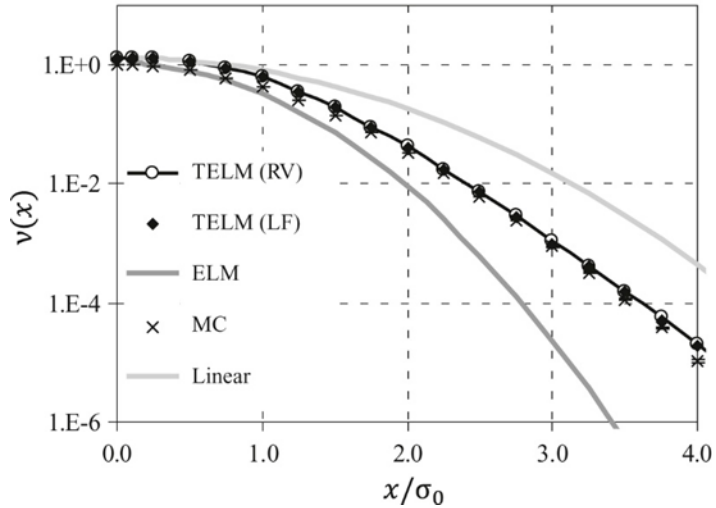

Mean upcrossing rate

Example – Hysteretic oscillator subjected to band-limited white noise base excitation

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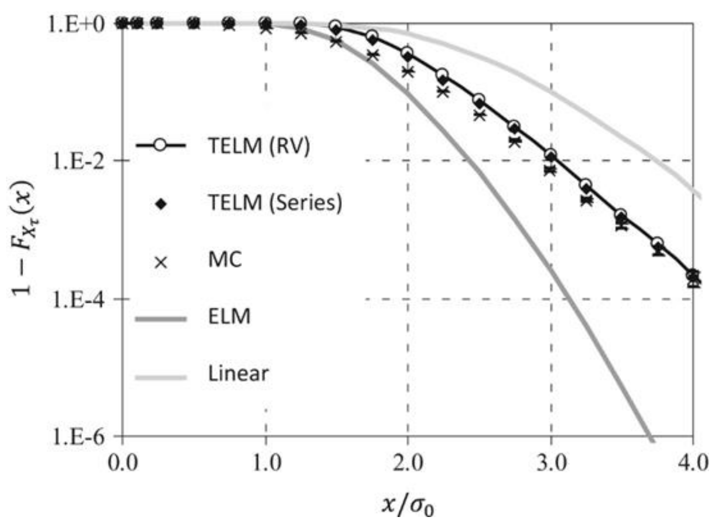
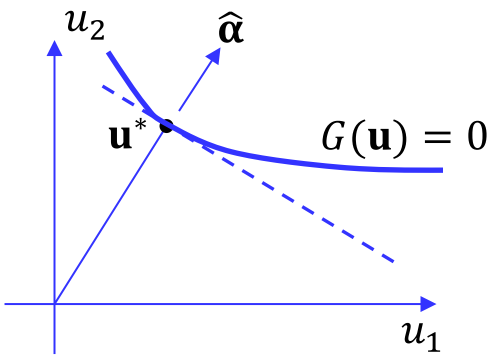


Complementary CDF of maximum absolute response over 10s duration

The nature of errors in FORM

- No measure of the error inherent in FORM is available.
- FORM tends to be more accurate for small probabilities (it is asymptotically exact as $\beta \to \infty$).
- Sources of error:
  - Nonlinearity in the limit-state surface.
  - Increasing dimension:

    While the design point has the highest probability density in the failure domain, with increasing dimension the volume at farther distances rapidly grows and the neighborhood of the design point is no more the dominant contributor to the failure probability integral.



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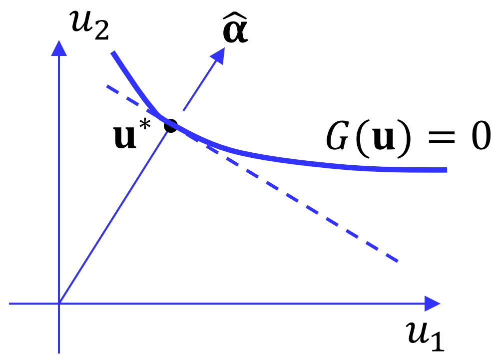
    However, in many high-dimension structural mechanics problems, the limit-state surface tends to be linear in many dimensions.



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- Sources of error:
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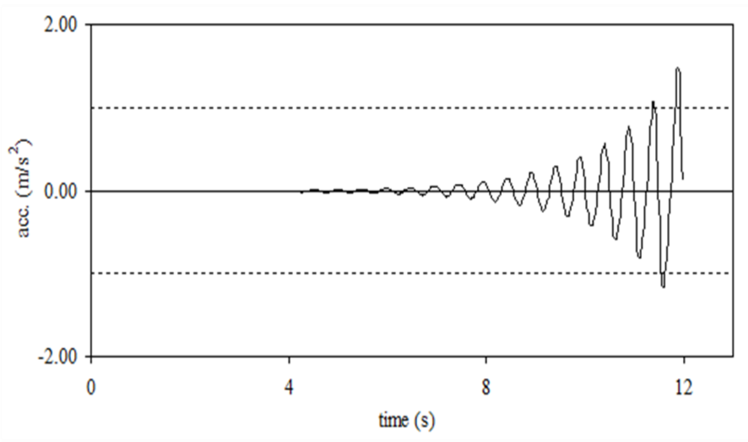
  - Increasing dimension:

    That is the case in FORM analysis of nonlinear stochastic dynamics problems, where the design point excitation is non-zero only near target time point of interest.



Design point excitation

Concluding remarks

- FORM provides an alternative for time-variant reliability analysis with the possibility of providing rich insight into the nature of the problem through the design point, importance measures, sensitivities, the equivalent linear system, etc.

- TELM is a non-parametric linearization method that can approximately determine the non-Gaussian distribution of nonlinear response.

- Many challenges persist:
  - Need response gradients;
  - Rapidly evolving nonstationary processes;
  - Non-differentiable hysteresis laws;
  - Degrading systems;
  - Effect of dimensionality;
  - etc.

Thank you!