Engineering Risk Analysis Group
TUM School of Engineering and Design
Technische Universität München

Sensitivity measures for time-variant reliability analysis

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Iason Papaioannou

Engineering Risk Analysis Group, TU München

E-mail: iason.papaioannou@tum.de

Overview

- Time-invariant reliability analysis
- Global sensitivity analysis for time-invariant problems
- Time-variant reliability analysis
- Global time-variant reliability sensitivity: Demonstration through a time-deteriorating system

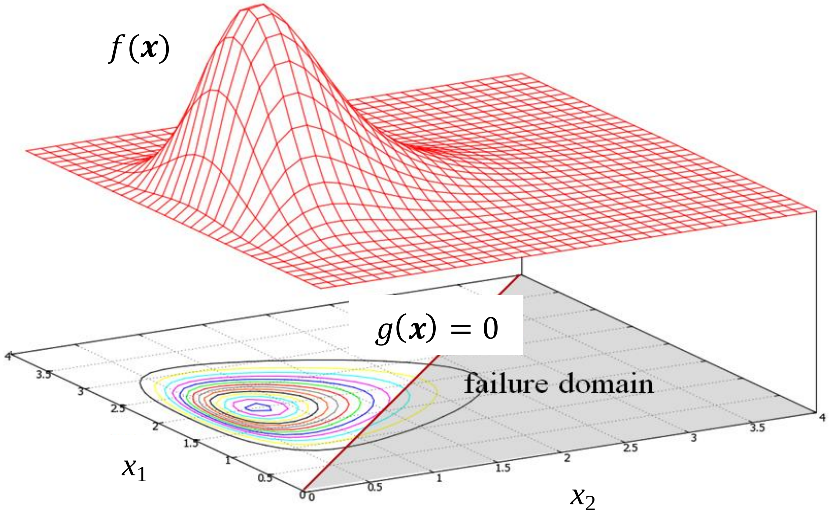
Time-invariant structural reliability problem

Vector of input random variables $\boldsymbol{X} = [X_1; X_2; \dots; X_d]$ with joint PDF $f(\boldsymbol{x})$

Limit-state function $g(\boldsymbol{x})$; Failure event $F = \{g(\boldsymbol{X}) \leq 0\}$

Probability of failure:

$$p_F := \mathbb{P}(F) = \int_{g(\boldsymbol{x}) \leq 0} f(\boldsymbol{x}) d\boldsymbol{x} = \mathbb{E}\left[\mathrm{I}(g(\boldsymbol{X}) \leq 0)\right]$$



Reliability sensitivity analysis

- Gradient-based sensitivity analysis: How does a change in the (deterministic) input parameters influence p_F? [Wu 1994; Jensen et al. 2009; Papaioannou et al. 2013, 2018]
- Variance-based/global sensitivity analysis: How does the variability of the input random variables influence p_F? [Li et al. 2012; Wei et al. 2012; Ehre et al. 2020]
- Decision-theoretic sensitivity analysis: How does the reduction of uncertainty of the input random variables influence the optimality of an engineering decision? [Straub et al. 2022]

Variance-based sensitivity analysis

Consider a d-dimensional independent random vector $\boldsymbol{X}$ and a function $Q = h(\boldsymbol{X})$.

ANOVA representation:

$$h(\boldsymbol{X}) = h_0 + \sum_{i=1}^{d} h_i(X_i) + \sum_{1 \le i < j \le d} h_{ij}\left(X_i, X_j\right) + \cdots + h_{1\ldots d}(X_1, \ldots, X_d)$$

The ANOVA representation exists and is unique provided that for any subset $\boldsymbol{X_v} \subseteq \boldsymbol{X}$ and any $i \in \boldsymbol{v}$

$$\mathbb{E}\left[h_{\boldsymbol{v}}(\boldsymbol{X_v}) | \boldsymbol{X}_{\boldsymbol{v} \setminus i}\right] = \int_{-\infty}^{\infty} h_{\boldsymbol{v}}(\boldsymbol{x_v})\, f_i(x_i) dx_i = 0$$

Variance decomposition:

$$\mathbb{V}(Q) = \sum_{i=1}^{d} V_i + \sum_{i<j}^{d} V_{ij} + \cdots + V_{1\ldots d}$$

Variance-based sensitivity indices

First-order (Sobol') indices:

$$S_i = \frac{V_i}{\mathbb{V}(Q)} = \frac{\mathbb{V}(\mathbb{E}[Q|X_i])}{\mathbb{V}(Q)}$$

Total effect indices

$$S_i^T = \frac{V_i + \sum_{j \neq i}^{d} V_{ij} + \cdots + V_{1\ldots d}}{\mathbb{V}(Q)} = 1 - \frac{\mathbb{V}(\mathbb{E}[Q|\boldsymbol{X}_{\sim i}])}{\mathbb{V}(Q)}$$

It holds: $0 \leq S_i \leq S_i^T \leq 1$

Note: The first-order index can be interpreted as decision-oriented sensitivity (expected value of partial perfect information) for a quadratic loss function

Variance-based reliability sensitivities [Wei et al. 2012]

Failure event $F = \{g(\boldsymbol{X}) \leq 0\}$

Quantity of interest: $Z = \mathrm{I}(g(\boldsymbol{X}) \leq 0)$

It is $\mathbb{E}[Z] = p_F, \mathbb{V}[Z] = p_F(1 - p_F)$

First-order indices:

$$S_{F,i} = \frac{\mathbb{V}(\mathbb{E}[Z|X_i])}{\mathbb{V}(Z)} = \frac{\mathbb{V}(\mathbb{P}[F|X_i])}{p_F(1 - p_F)}$$

Total-effect indices:

$$S_{F,i}^T = 1 - \frac{\mathbb{V}(\mathbb{E}[Z|\boldsymbol{X}_{\sim i}])}{\mathbb{V}(Z)} = 1 - \frac{\mathbb{V}(\mathbb{P}[F|\boldsymbol{X}_{\sim i}])}{p_F(1 - p_F)}$$

Other variance-based reliability sensitivities (I)

Define: $\boldsymbol{X}_{\boldsymbol{v}} = \{X_i, i \in \boldsymbol{v}\}$ with $\boldsymbol{v} \in \mathcal{P}(\{1, \dots, d\})$

Closed Sobol' index associated with $\boldsymbol{X}_{\boldsymbol{v}}$:

$$S_{F,\boldsymbol{v}} = \frac{\mathbb{V}(\mathbb{P}[F|\boldsymbol{X}_{\boldsymbol{v}}])}{p_F(1-p_F)}$$

Total-effect index associated with $\boldsymbol{X}_{\boldsymbol{v}}$:

$$S_{F,\boldsymbol{v}}^T = 1 - \frac{\mathbb{V}(\mathbb{P}[F|\boldsymbol{X}_{\sim\boldsymbol{v}}])}{p_F(1-p_F)}$$

Closed Sobol' indices are particularly relevant if $\boldsymbol{X}_{\boldsymbol{v}}$ represents the effect of a single physical quantity

Other variance-based reliability sensitivities (II)

Uncertainty separation: $\boldsymbol{X} = [\boldsymbol{X}_A; \boldsymbol{X}_B]$

Failure probability conditional on $\boldsymbol{X}_B$: $P_F(\boldsymbol{x}_B) := \mathbb{P}(F|\boldsymbol{X}_B = \boldsymbol{x}_B) = \mathbb{E}\left[\mathrm{I}(g(\boldsymbol{X}) \leq 0)|\boldsymbol{X}_B = \boldsymbol{x}_B\right]$

Sobol' indices of P_F [Morio 2011; Wang et al. 2013]:

$$S_{P_F,i} = \frac{\mathbb{V}(\mathbb{E}[P_F|X_i])}{\mathbb{V}(P_F)}$$

Sobol' indices of $\log P_F$ [Ehre et al. 2020]:

$$S_{\log P_F,i} = \frac{\mathbb{V}(\mathbb{E}[\log P_F\,|X_i])}{\mathbb{V}(\log P_F)}$$

Interpretation of reliability sensitivities

- First-order indices can be used for factor prioritization, to determine which random variable if learned will have the largest impact on the value of p_F
- Total-effect indices can be used for factor fixing, to determine the random variables with $S_{F,i}^T \approx 0$, which if fixed will not impact the prediction of p_F

Estimation of reliability sensitivities

- Peek-freeze estimators combined with importance sampling [Wei et al. 2012]
- Estimation with failure samples [Perrin & Defaux 2019; Li, Papaioannou & Straub 2019]
- Estimation with FORM [Papaioannnou & Straub 2021; 2024]
- Surrogate modeling-based estimation [Wang et al. 2013; Ehre et al. 2020]

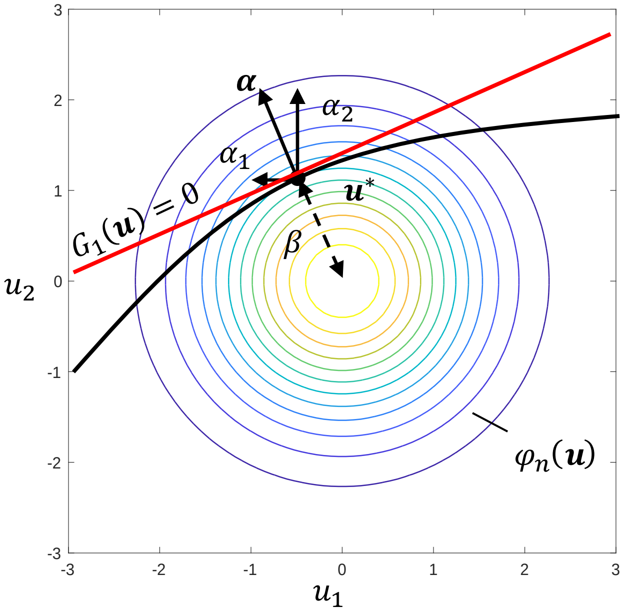
FORM alpha-factors

Directional cosines of the most likely failure point $\boldsymbol{u}^*$ in an equivalent standard normal space $\boldsymbol{U} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I})$:

$$\boldsymbol{\alpha} = \frac{\boldsymbol{u}^*}{\beta}$$

The squared factors α_i^2 are the Sobol' indices of the linearized LSF $G_1(\boldsymbol{U})$ at the design point:

$$S_{G_1,i} = \frac{\mathbb{V}(\mathrm{E}[G_1|U_i])}{\mathbb{V}(G_1)} = \alpha_i^2$$



Variance-based sensitivities with FORM [Papaioannou & Straub 2021]

First-order index:

$$S_{F,i} \approx \frac{1}{p_{F_1}\left(1-p_{F_1}\right)} \int_0^{\alpha_i^2} \varphi_2(-\beta,-\beta,r)\,dr$$

Total effect index:

$$S_{F,i}^T \approx \frac{1}{p_{F_1}\left(1-p_{F_1}\right)} \int_{1-\alpha_i^2}^{1} \varphi_2(-\beta,-\beta,r)\,dr$$

with

$$\varphi_2(-\beta,-\beta,r) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\left(-\frac{\beta^2}{1+r}\right)$$

FORM approximation of sensitivities for system reliability problems are given in [Papaioannou & Straub 2024]

Time-variant reliability analysis

Vector of input random variables $\boldsymbol{X} = [X_1; \dots; X_d]$ and random processes $\boldsymbol{Y}(t) = [Y_1(t); \dots; Y_k(t)]$

Limit-state function $g\big(t, \boldsymbol{x}, \boldsymbol{y}(t)\big)$; Instantaneous failure event $F_t = \left\{g\big(t, \boldsymbol{X}, \boldsymbol{Y}(t)\big) \leq 0\right\}$

Instantaneous probability of failure:

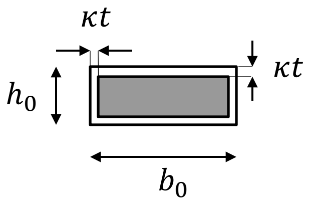
$$p_{F,i}(t) := \mathbb{P}(F_t) = \mathbb{E}\left[\mathrm{I}\Big(g\big(t, \boldsymbol{X}, \boldsymbol{Y}(t)\big) \leq 0\Big)\right]$$

Cumulative probability of failure in the interval $[0, T]$:

$$p_{F,c}(0, T) := \mathbb{P}\Big(\exists t \in [0, T] : g\big(t, \boldsymbol{X}, \boldsymbol{Y}(t)\big) \leq 0\Big) = \mathbb{E}\left[\mathrm{I}\left(\min_{0 \leq t \leq T} g\big(t, \boldsymbol{X}, \boldsymbol{Y}(t)\big) \leq 0\right)\right]$$

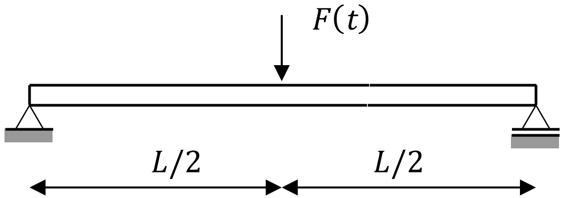
Corroded beam under random loading [Andrieu-Renaud et al. 2004]

Limit state function:



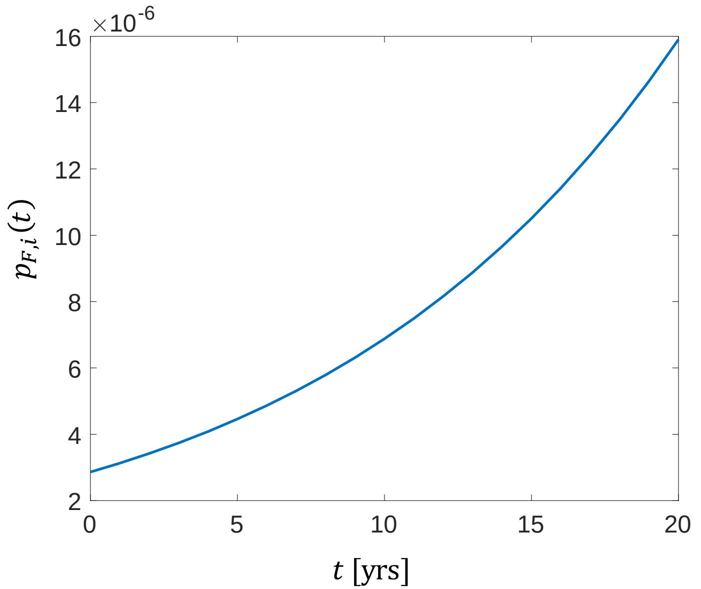
$$g\big(t, \boldsymbol{X}, \boldsymbol{Y}(t)\big) = M_u(t) - M(t) = \frac{(b_0 - 2\kappa t)(h_0 - 2\kappa t)^2}{4} f_y - \left(\frac{F(t)L}{4} + \frac{\rho_{st} b_0 h_0 L^2}{8}\right)$$

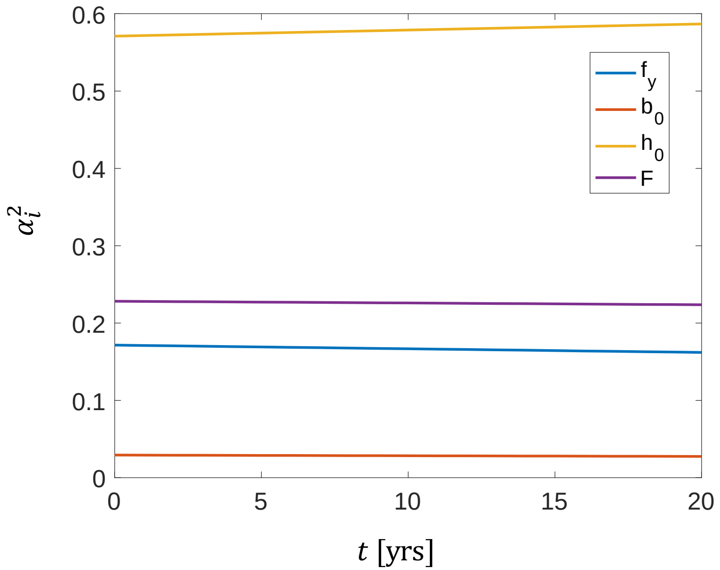
Parameter	Distribution	Mean	CoV	Auto-correlation
f_y	Lognormal	$240\ \mathrm{Mpa}$	$10\ \%$	NA
b_0	Lognormal	$0.2\ \mathrm{m}$	$5\ \%$	NA
h_0	Lognormal	$0.04\ \mathrm{m}$	$10\ \%$	NA
F	Gaussian	$3500\ \mathrm{N}$	$20\ \%$	$\exp\left(-\left(\frac{t_2 - t_1}{\lambda}\right)^2\right);\ \lambda = 1\mathrm{mth}$



Instantaneous probability of failure

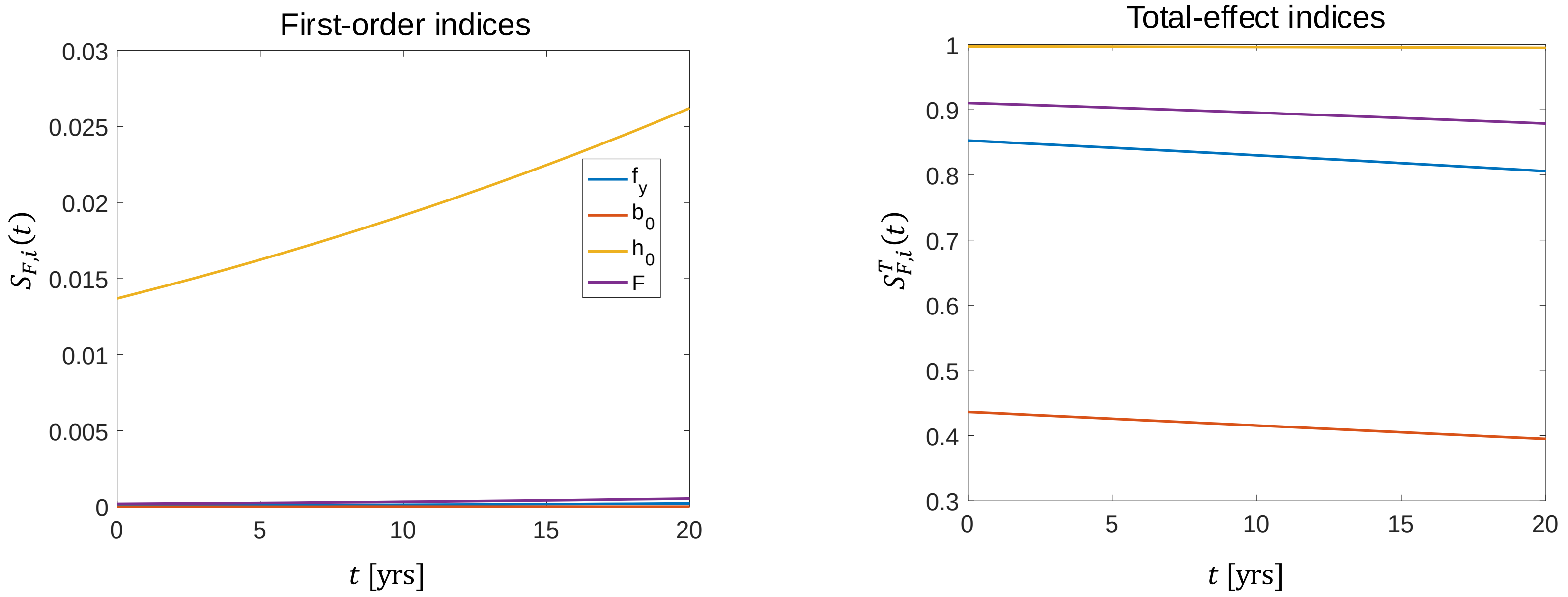
FORM estimates of $p_{F,i}(t) = \mathbb{E}\left[\mathrm{I}\left(g\left(t, f_y, b_0, h_0, F(t)\right) \leq 0\right)\right]$ and alpha-factors





Instantaneous probability of failure (II)

FORM estimates of Sobol' indices of $Z(t) = \mathrm{I}\left(g\left(t, f_y, b_0, h_0, F(t)\right) \leq 0\right)$



Interval probability of failure

Define interval failure events:

$$F_j^* = \left\{ \min_{\tau \in (t_{j-1}, t_j]} g\big(\tau, \boldsymbol{X}, \boldsymbol{Y}(\tau)\big) \le 0 \right\}, j = 1, \dots, m \text{ with } t_j = j\Delta t$$

Interval failure probability: $p_{F,j}^* = \mathbb{P}\big(F_j^*\big)$

Corroded beam example:

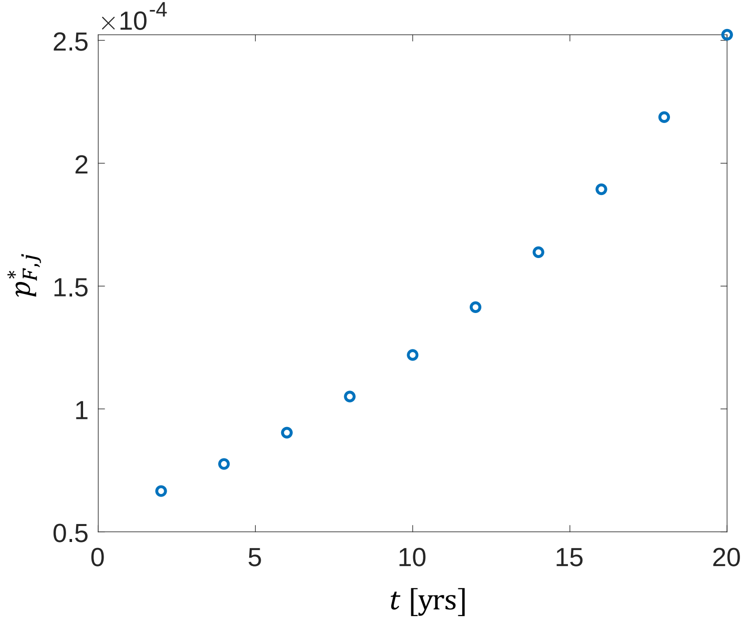
Interval failure probability:

$$p_{F,j}^* \approx \mathbb{P}\big(g\big(t_j, f_y, b_0, h_0, F_{\max,j}\big) \le 0\big)$$

with $F_{\max,j} = \max_{\tau \in (t_{j-1}, t_j]} F(\tau)$

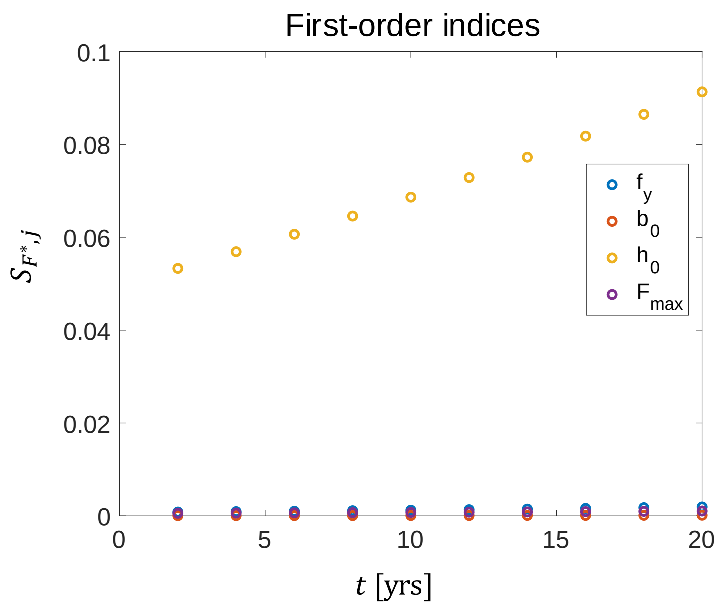
$F_{\max}$ can be approximated by a Gumbel random variable

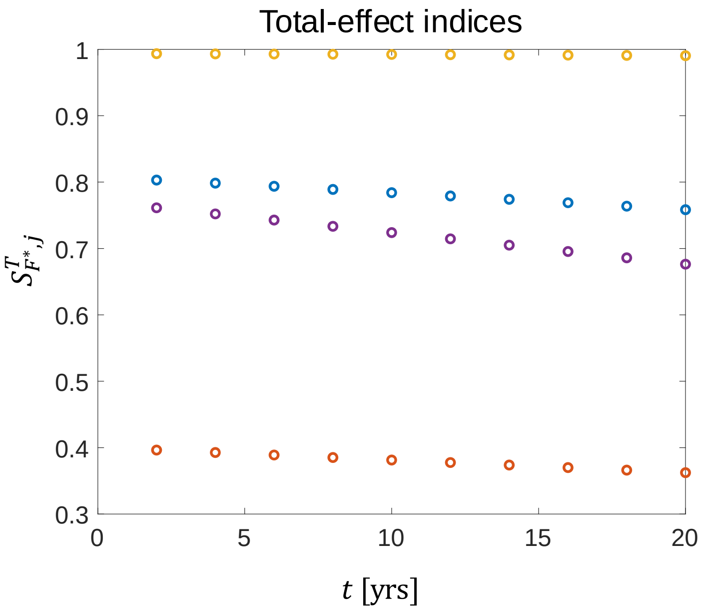
FORM approximation of $p_{F,j}^*$ with $\Delta t = 2$ yrs



Interval probability of failure (II)

FORM estimates of Sobol' indices of $Z_j^* = \mathrm{I}\left(g\left(t_j, f_y, b_0, h_0, F_{\max,j}\right) \leq 0\right)$





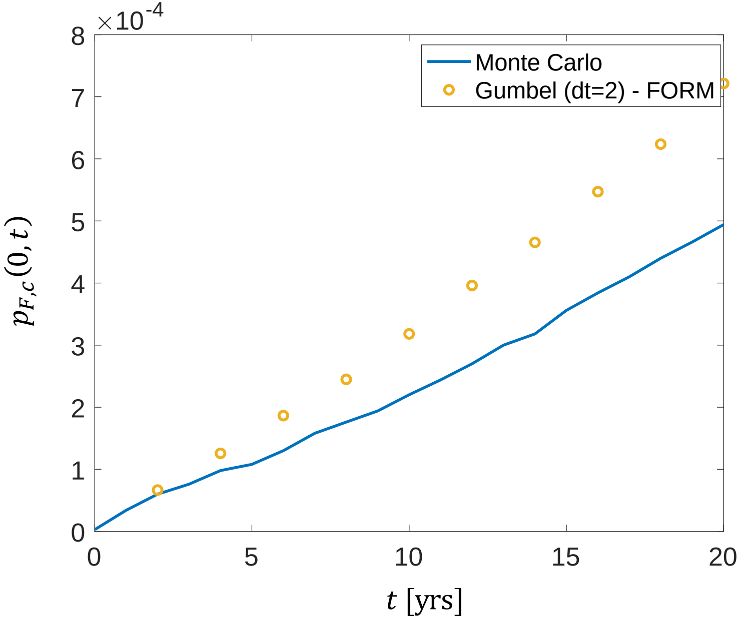
Cumulative probability of failure: Interval approach

Cumulative probability of failure in the interval $[0, t_i]$:

$$p_{F,c}(0, t_i) = \mathbb{P}(F_1^* \cup F_2^* \cup \cdots \cup F_i^*)$$

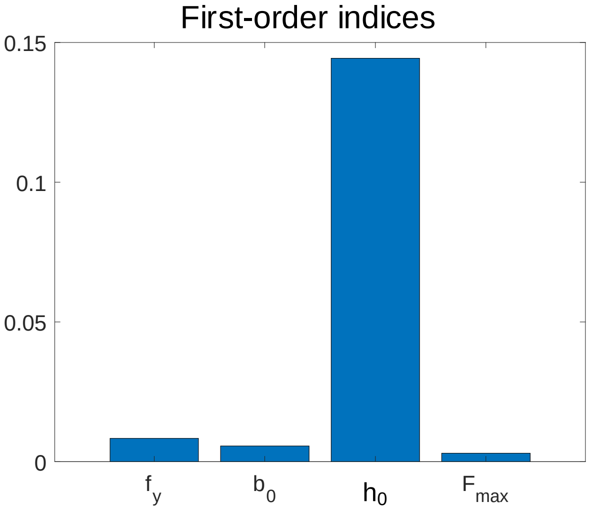
Corroded beam example:
Assume that the maximum loads over intervals $\Delta t = 2\text{yrs}$ are statistically independent

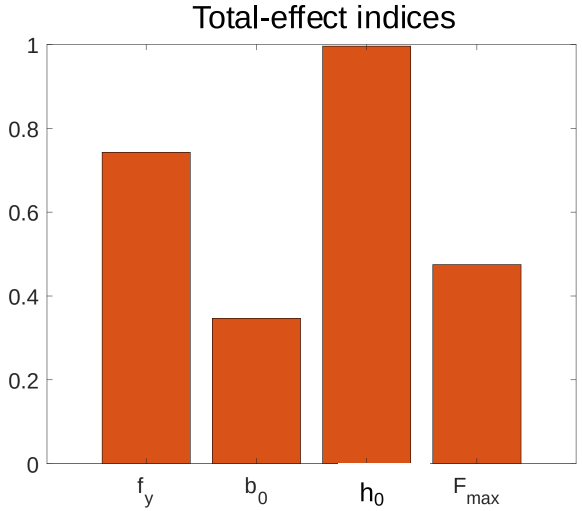
The cumulative probability of failure can be approximated with FORM for system problems [Hohenbichler & Rackwitz 1982]



Cumulative probability of failure: Interval approach (II)

FORM estimates of Sobol' indices of $Z_{\text{ser}} = 1 - \prod_{i=1}^{m}\left(1 - Z_j^*\right)$ for the full time period $[0{,}20\ \text{yrs}]$





Note: The closed Sobol' index of the combined effect of all (independent) load variables $F_{\max,j}$ is shown

Cumulative probability of failure: Outcrossing approach

Conditional cumulative probability of failure given system parameters $\boldsymbol{X}_R$:

$$P_{F,[0,T]}(\boldsymbol{x}_R) := \mathbb{P}\big(\exists t \in [0,T]: g\big(t, \boldsymbol{X}_R, \boldsymbol{Y}_S(t)\big) \leq 0 | \boldsymbol{X}_R = \boldsymbol{x}_R\big)$$

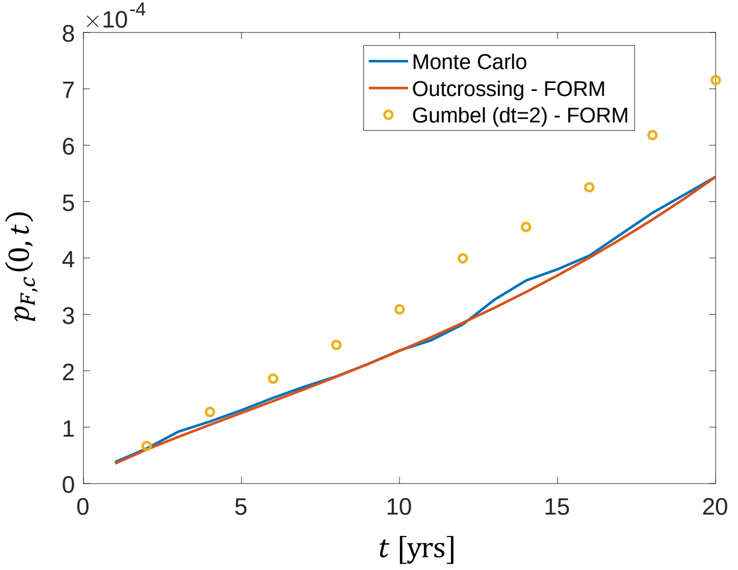
Corroded beam example: $P_{F,[0,T]}(\boldsymbol{x}_R)$, with $\boldsymbol{X}_R = [f_y, b_0, h_0]$ can be estimated efficiently based on outcrossing theory

Cumulative probability of failure: $p_{F,c}(0,T) = \mathbb{E}\big[P_{F,[0,T]}(\boldsymbol{X}_R)\big]$

Auxiliary limit-state function [Wen & Chen 1987]:

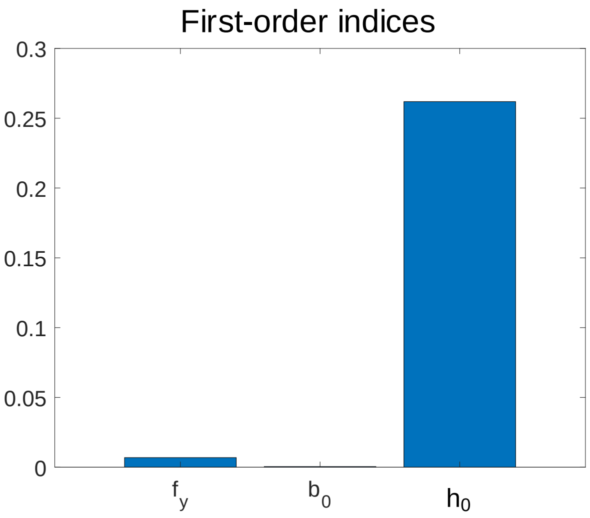
$$g(\boldsymbol{x}_R, z) = z - P_{F,[0,T]}(\boldsymbol{x}_R)$$

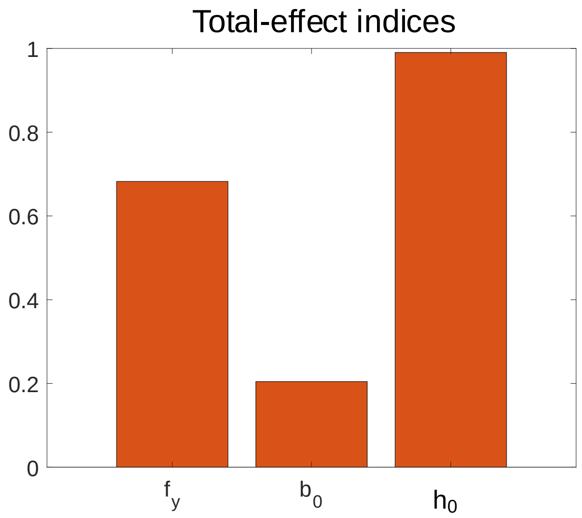
where z is the outcome of a standard uniform random variable



Cumulative probability of failure: Outcrossing approach (II)

FORM estimates of Sobol' indices of $P_{F,[0,T]}(f_y, b_0, h_0)$ for the full time period $[0{,}20\ \mathrm{yrs}]$





Summary

- Sensitivity measures for time-variant reliability analysis
- Application of Sobol’ sensitivities to different time-variant reliability scenarios
- Estimation of time-variant reliability sensitivities with FORM
- Demonstrated the behavior of the sensitivities with the reliability of a deteriorating structure

References

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