Tutorial on time-variant reliability analysis

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Time-variant reliability: definitions

At time t, the structure can be characterized by its resistance (or, capacity), $R(t)$, and the load (or, demand) on the structure, $S(t)$. $R(t)$ and $S(t)$ are random variables.

- One can define a point-in-time failure event as

$$F^*(t) = \{M(t) \leq 0\} = \{R(t) \leq S(t)\}$$

  where $M(t) = R(t) - S(t)$ is the safety margin.

- More generally, the structure can be modelled by a limit state function $g(\mathbf{Y}, t)$, where $\mathbf{Y}$ is comprised of all random variables and random processes in the problem. Then

$$F^*(t) = \{g(\mathbf{Y}, t) \leq 0\} \, .$$

Time-variant reliability: definitions (2)

To compute the time-variant reliability, one must consider the random processes $\{R(\tau)\}_{\tau\in[0,T]}$ and $\{S(\tau)\}_{\tau\in[0,T]}$, and account for all point-in-time failure events up to time T.

- The failure event for a given time duration $[0,T]$ is defined as:

$$F(T) = \{\exists\tau \in [0,T]: R(\tau) \leq S(\tau)\}$$

- The time-variant structural reliability is given by:

$$\mathbb{L}(0,T) = \Pr\{R(t) > S(t), \forall t \in [0,T]\} = 1 - \Pr\{F(T)\}.$$

- Using the more general definition, the probability of a failure up to time T is given by:

$$P_F(0,T) = \Pr\{F(T)\} = \Pr\{\min_{0\leq\tau\leq T} g(\mathbf{Y},\tau) \leq 0\} \;.$$

Time-dependent reliability: basic approaches

- **Time-integrated approach**: the whole service period $[0, T]$ of the structure is considered as a single time unit. The reliability is computed based on statistical properties of the random variables that relate to the whole service period.

- **Discretized approach**: Shorter time intervals, such as a year or duration of a storm, are considered. The reliability within each time unit is estimated based on extreme value theory. Failure probability over the whole service period $[0, T]$ is then determined (approximately) from the interval failure probabilities.

- **Out-crossing theory based approach**: The structural failure event is viewed as an outcrossing event of a random process. "Outcrossing" implies that the safety margin process $M(t) = R(t) - S(t)$, or the random process $g(\mathbf{Y}, \tau)$, becomes zero or less in the period $[0, T]$. We estimate the first-passage probability, i.e., the probability that $M(t) \leq 0$ occurs during $[0, T]$ using random process theory.

Time-integrated approach

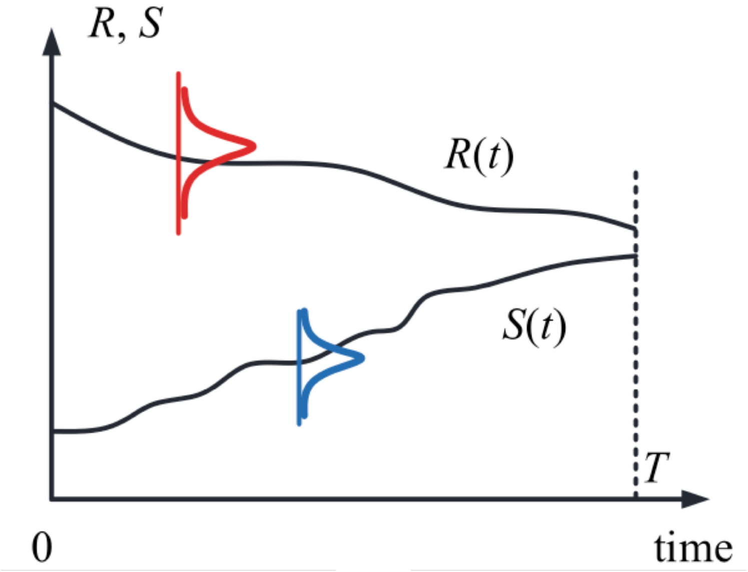
We consider a case in which every realization of $S(t)$ is non-decreasing and every realization of $R(t)$ is non-increasing.

In this case, it is reasonable to compare the R_{min} with S_{max}, both occurring at time $t = T$.

An instantaneous estimate of the failure probability at time t is given by:

$$\Pr\{F^*(t)\} = \Pr\{R(t) < S(t)\}$$

In this case, $P_F(0,T) = \Pr\{F^*(T)\}$. This probability can be computed by time-invariant reliability methods.



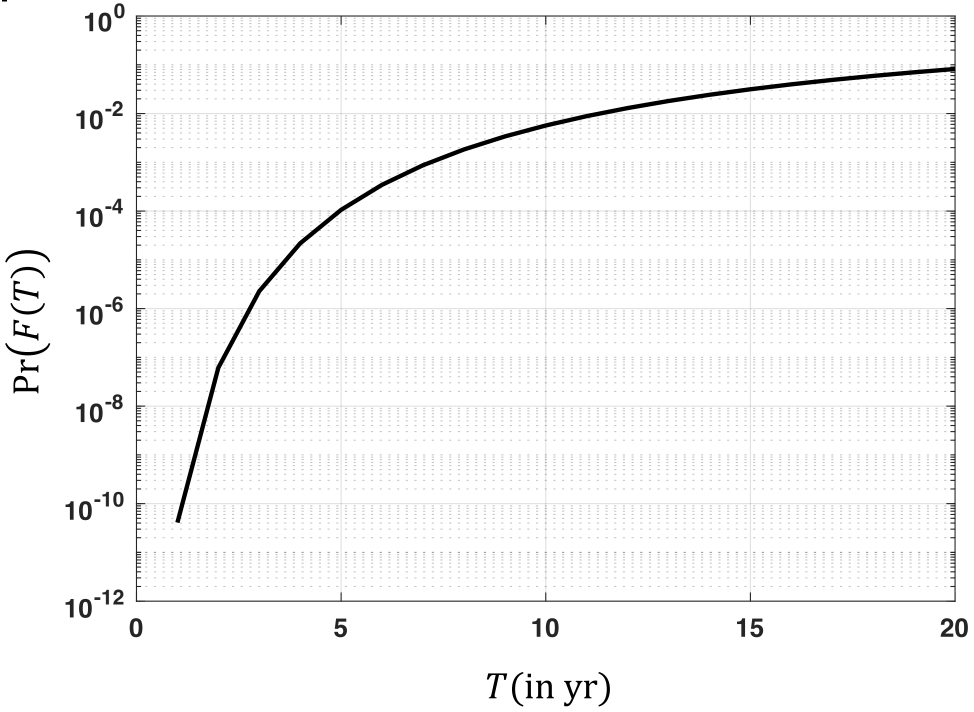
This approach is applicable to the general case, if $g(\mathbf{Y}, t)$ is monotonically non-increasing, e.g., $g(\mathbf{Y}, t) = D_{cr} - h_d(\mathbf{Y}, t)$ where $h_d(\mathbf{Y}, t)$ is a deterioration model and D_{cr} is the deterioration limit.

Image Source: Wang C. Structural reliability and time-dependent reliability. 2021 Springer series in reliability engineering, Springer Cham. DOI https://doi.org/10.1007/978-3-030-62505-4 .

Example 1

Consider a steel plate subjected to corrosion. Failure occurs when the corrosion loss exceeds the plate thickness w.

The limit state function is $g(\mathbf{Y}, t) = w - A(t - C)$, where $\mathbf{Y} = [A, C]$.
A represents the corrosion rate and C denotes the coating life.



A, C: Lognormal random variables, independent

$\mu_A = 0.6\ \text{mm/yr}$ (mean), $\sigma_A = 0.5\ \text{mm/yr}$ (standard deviation)

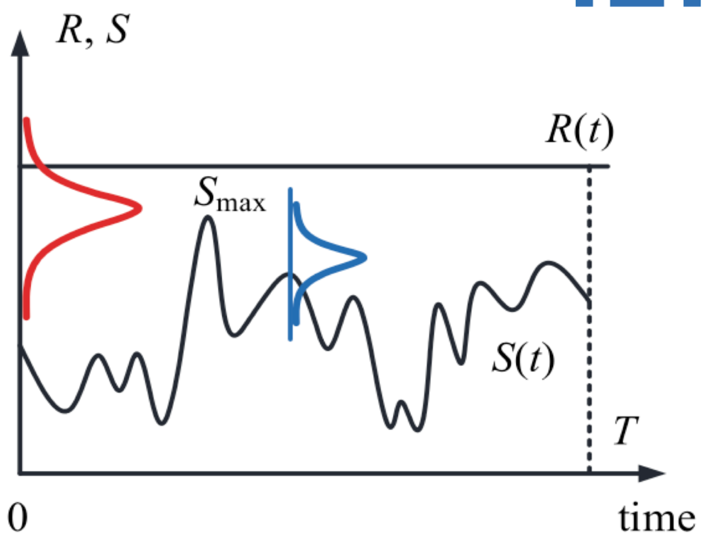
$\mu_C = 5.0\ \text{yr}$, $\sigma_C = 5.0\ \text{yr}$

$$\Pr\big(F(T)\big) = \Pr\big(F^*(T)\big) = \Pr\{g(\mathbf{Y}, T) \leq 0\}$$

Exact solution possible through numerical intergation

Reference: Straub D., Schneider R., Bismut E. and Kim H.Y. Reliability analysis of deteriorating structural systems. Structural Safety, Volume 82, 2020, Article 101877.

Time-integrated approach (2)



The monotonicity assumption on $S(t)$ may not hold in general.

Consider the R-S problem. If $R(t) \equiv R$:

$$P_F(0,T) = \Pr(R < S_{max}),$$

where $S_{max} = \max\{S(t) | t \in [0,T]\}$.

If a suitable model for the CDF $F_{S_{max}}(\cdot)$ of S_{max} is available, then $P_F(0,T)$ can be evaluated by time-invariant reliability methods.

If $R(t)$ is time-variant, or, more generally, if $g(\mathbf{Y},t)$ shows random fluctuations in time, advanced methods based on out-crossing theory or Monte Carlo simulation are typically required to evaluate $P_F(0,T)$.

Image Source: Wang C. Structural reliability and time-dependent reliability. 2021 Springer series in reliability engineering, Springer Cham. DOI https://doi.org/10.1007/978-3-030-62505-4 .

Discretized approach

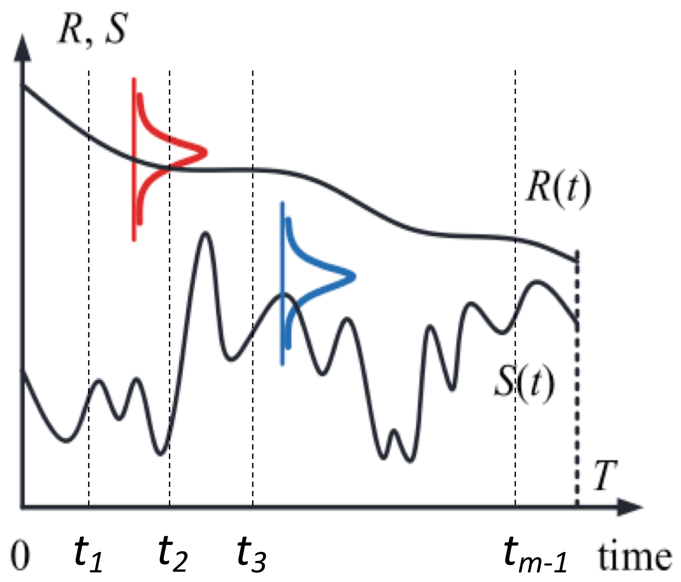
- Divide the whole service period $[0, T]$ into multiple time units $0 = t_0 < t_1 < \cdots < t_m = T$
- A common choice is yearly intervals, but a finer discretization is adopted if the service life is short
- Define interval failure events $F_j^* = \left\{\min_{\tau \in (t_{j-1}, t_j]} R(\boldsymbol{Y}_R, \tau) - S(\tau) \le 0\right\}; j = 1, \ldots, m$; random variables $\boldsymbol{Y}_R$ that determine resistance are separable from those that determine the load effects.
- $R(t)$ is monotonically non-increasing.
- Interval failure probability:

  $\Pr(F_j^*) \approx \Pr\left(R(\boldsymbol{Y}_R, t_j) \le S_{max,j}\right)$ (conservative)

  $S_{max,j} = \max\{S(t) | t \in (t_{j-1}, t_j]\}$.

  maximum error $\Pr\left(R(\boldsymbol{Y}_R, t_j) \le S_{max,j}\right) - \Pr\left(R(\boldsymbol{Y}_R, t_{j-1}) \le S_{max,j}\right)$

- $\Pr(F_j^*)$ can be estimated by time-invariant reliability analysis.



Discretized approach (2)

- In the general case $\mathbf{Y} = \{\boldsymbol{Y}_R, \boldsymbol{S}_{ti}, \boldsymbol{S}(t)\}$, where $\boldsymbol{S}_{ti}$ is the vector of time-invariant load effects and $\boldsymbol{S}(t)$ is the vector of time-variant load effects.
- Interval failure events: $F_j^* = \left\{\min_{\tau\in(t_{j-1},t_j]} g(\boldsymbol{Y}_R, \boldsymbol{S}_{ti}, \boldsymbol{S}(\tau), \tau) \leq 0\right\}$.
- Interval failure probability: $\Pr\left(F_j^*\right) \approx \Pr\left(g\left(\boldsymbol{Y}_R, \boldsymbol{S}_{ti}, \boldsymbol{S}_{max,j}, t_j\right) \leq 0\right)$.
- The target failure event is a union of the interval failure events: $F(T) = \bigcup_{j=1}^{m} F_j^*$.
- Computation of $\Pr\left(F(T)\right)$ requires accounting for the dependence between the interval failure events.
- Exact computation of $\Pr\left(F(T)\right)$ requires solving a series system reliability problem.
- Series system bounds for $\Pr\left(F(T)\right)$:

$$\max_{j=1,\dots,m} \Pr\left(F_j^*\right) \leq \Pr\left(F(T)\right) \leq \sum_{j=1}^{m} \Pr\left(F_j^*\right)$$

Monte Carlo simulation

- Simplest and most robust strategy to compute $\Pr\big(F(T)\big) = \Pr\big(\bigcup_{j=1}^{m} F_j^*\big)$
- Generate samples $\boldsymbol{Y}_R^{(k)}, \boldsymbol{S}_{ti}^{(k)}, \boldsymbol{S}_{max,1}^{(k)}, \dots, \boldsymbol{S}_{max,m}^{(k)}, k = 1, \dots, N$
- Estimate of the failure probability:

$$\Pr\big(F(T)\big) \approx \frac{1}{N}\sum_{k=1}^{N} \mathbb{I}\left\{\left[\min_{j=1,\dots,m} g\left(\boldsymbol{Y}_R^{(k)}, \boldsymbol{S}_{ti}^{(k)}, \boldsymbol{S}_{max,j}^{(k)}, t_j\right)\right] \leq 0\right\}$$

- Coefficient of variation of the estimate: $\delta = \sqrt{\frac{\left(1-\Pr\left(F(T)\right)\right)}{N\Pr\left(F(T)\right)}}$
- Inefficient when computing small failure probabilities. Approximately, 10^8 samples are required to estimate a failure probability of $\Pr\big(F(T)\big) = 10^{-6}$ with a coefficient of variation of $\delta = 10\%$. Infeasible for problems with computationally expensive limit states.
- If the point-in-time failure probability is desirable, e.g., in the case of a monotonically decreasing limit state, it can be estimated based on this approach.

Evaluation of $\Pr(F_j^*)$ - First order reliability method

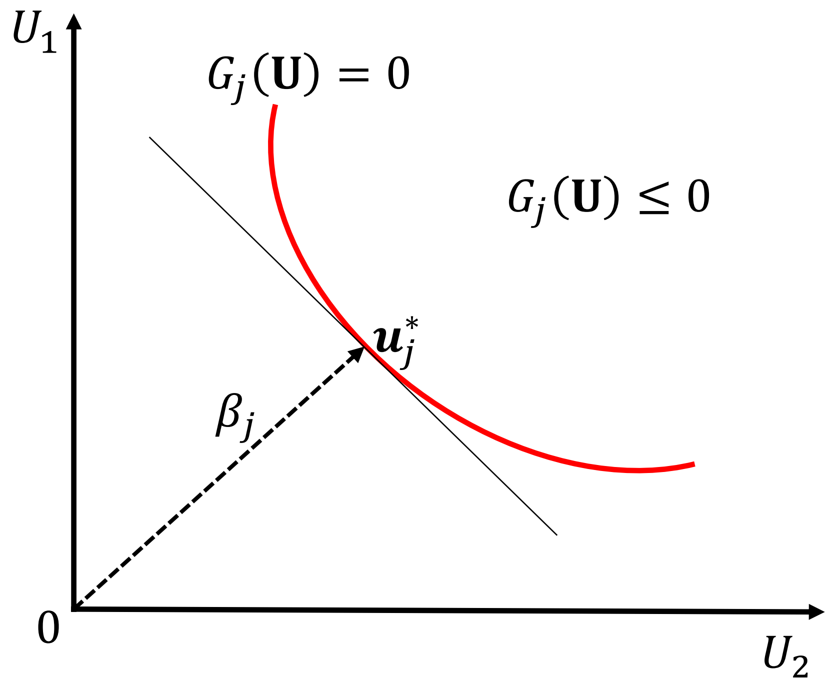
- Consider the interval failure probability $\Pr(F_j^*) = \Pr\{g_j(\mathbf{Y}) \leq 0\}$, where $g_j(\mathbf{Y}) = g(\boldsymbol{Y}_R, \boldsymbol{S}_{ti}, \boldsymbol{S}_{max,j}, t_j)$ and $\mathbf{Y} = [\boldsymbol{Y}_R, \boldsymbol{S}_{ti}, \boldsymbol{S}_{max,j}]$.
- Random vector $\mathbf{Y}$ is transformed to uncorrelated standard normal random vector $\mathbf{U} = \mathbb{T}(\mathbf{Y})$ through iso-probabilistic transformations.
- The limit state is expressed in terms of $\mathbf{U}$: $G_j(\mathbf{U}) = g_j(\mathbb{T}^{-1}(\mathbf{U}))$
- FORM uses the design point $\boldsymbol{u}_j^*$ in the $\mathbf{U}$ - space:

$$\boldsymbol{u}_j^* = \operatorname{argmin}\|\mathbf{u}\|$$

s.t. $G_j(\mathbf{U}) \leq 0$.

where $\|\cdot\|$ is the Euclidean norm.

- $\Pr(F_j^*) \approx \Phi(-\beta_j)$ where, $\beta_j = \|\boldsymbol{u}_j^*\|$ is the reliability index.
- $\boldsymbol{u}_j^*$ can be used as the starting point of numerical optimization to determine $\boldsymbol{u}_{j+1}^*$.



Evaluation of $\Pr(F_j^*)$ - First order reliability method (2)

FORM approximation linearizes the limit state function $G_j(\mathbf{U})$ at $\mathbf{U} = \boldsymbol{u}_j^*$, i.e.,

$$\Pr(F_j^*) = \Pr\{G_j(\mathbf{U}) \le 0\} \approx \Pr\{G_j^l(\mathbf{U}) \le 0\}$$

where,

$$G_j(\mathbf{U}) \approx G_j^l(\mathbf{U}) = G_j(\boldsymbol{u}_j^*) + \langle \nabla G_j(\boldsymbol{u}_j^*), (\mathbf{U} - \boldsymbol{u}_j^*) \rangle = \langle \nabla G_j(\boldsymbol{u}_j^*), (\mathbf{U} - \boldsymbol{u}_j^*) \rangle$$

- $G_j^l(\mathbf{U})$ is a normal random variable:
  - $\mu_j^l = \mathrm{E}[G_j^l(\mathbf{U})] = -\langle \nabla G_j(\boldsymbol{u}_j^*), \boldsymbol{u}_j^* \rangle$ and $\sigma_j^l = \sqrt{\mathrm{Var}[G_j^l(\mathbf{U})]} = \|\nabla G_j(\boldsymbol{u}_j^*)\|$
  - $M_j = \frac{G_j^l(\mathbf{U}) - \mu_j^l}{\sigma_j^l}$ is a standard normal random variable
  - If $\boldsymbol{\alpha}_j = \frac{\boldsymbol{u}_j^*}{\|\boldsymbol{u}_j^*\|}$, then $\frac{\mu_j^l}{\sigma_j^l} = \langle \boldsymbol{\alpha}_j, \boldsymbol{u}_j^* \rangle = \|\boldsymbol{u}_j^*\| = \beta_j$.
  - $\Pr\{G_j^l(\mathbf{U}) \le 0\} = \Pr\{M_j \le -\beta_j\} = \Phi(-\beta_j)$

Evaluation of $\Pr(F(T)) = \Pr(\bigcup_{j=1}^{m} F_j^*)$ - extension of FORM

Recall that $F(T) = \bigcup_{j=1}^{m} F_j^*$

- $\Pr(\bigcup_{j=1}^{m} F_j^*) = 1 - \Pr(\bigcap_{j=1}^{m} F_j^{*c}) \approx 1 - \Pr(\bigcap_{j=1}^{m}\{M_j > -\beta_j\})$
- The random variables $M_1, \dots, M_m$ are mutually correlated and jointly Gaussian.
- Since $M_1, \dots, M_m$ have zero mean value, from symmetry we get

$$\Pr(F(T)) = 1 - \Pr(\bigcap_{j=1}^{m} F_j^{*c}) \approx 1 - \Phi_m(\boldsymbol{\beta}; \boldsymbol{\rho})$$

where $\Phi_m(\boldsymbol{\beta}; \boldsymbol{\rho})$ is the multivariate standard normal CDF with correlation matrix $\boldsymbol{\rho}$ evaluated at $\boldsymbol{\beta} = (\beta_1, \dots, \beta_m)$. The element $[\boldsymbol{\rho}]_{j,k}$ is the correlation coefficient between M_j and M_k and is given by $[\boldsymbol{\rho}]_{j,k} = \mathrm{E}[M_j M_k]$. $[\boldsymbol{\rho}]_{j,k}$ can be computed from the form sensitivities $\boldsymbol{\alpha}_j$ and $\boldsymbol{\alpha}_k$.

Evaluation of $\Pr(F_j^*)$ - Subset simulation

The probability $\Pr(F_j^*) = \Pr\{G_j(\mathbf{U}) \leq 0\}$ is formulated as a sequence of conditional probabilities:

$$\Pr\{G_j(\mathbf{U}) \leq 0\} = \prod_{k=1}^{L} \Pr\{G_j(\mathbf{U}) \leq b_k \mid G_j(\mathbf{U}) \leq b_{k-1}\} = \prod_{k=1}^{L} p_{k,k-1}$$

L is the number of subsets and the b_k are the intermediate thresholds with $\infty = b_0 \geq b_1 \geq \cdots \geq b_L = 0$.

- The intermediate thresholds are selected so that the conditional probabilities are large, typically $\rho = 0.1$.
- The probability $p_{k,k-1}$ can be estimated (accurately) by the standard Monte Carlo method:

$$p_{k,k-1} \approx \frac{1}{N} \sum_{s=1}^{N} \mathbb{I}\{G_j(\mathbf{U}^{(s)}) \leq b_k\}$$

  Samples $\mathbf{U}^{(s)}, s = 1, \ldots, N$ are distributed according to the PDF $f_{\mathbf{U},k-1}(\mathbf{u}) = \frac{\mathbb{I}[G_j(\mathbf{u}) \leq b_{k-1}] f_{\mathbf{U}}(\mathbf{u})}{\Pr\{G_j(\mathbf{u}) \leq b_{k-1}\}}$, where $f_{\mathbf{U}}(\mathbf{u})$ is the nominal PDF of $\mathbf{U}$. The samples are generated through the Markov chain Monte Carlo method.

- During implementation, the thresholds are selected through an adaptive procedure.

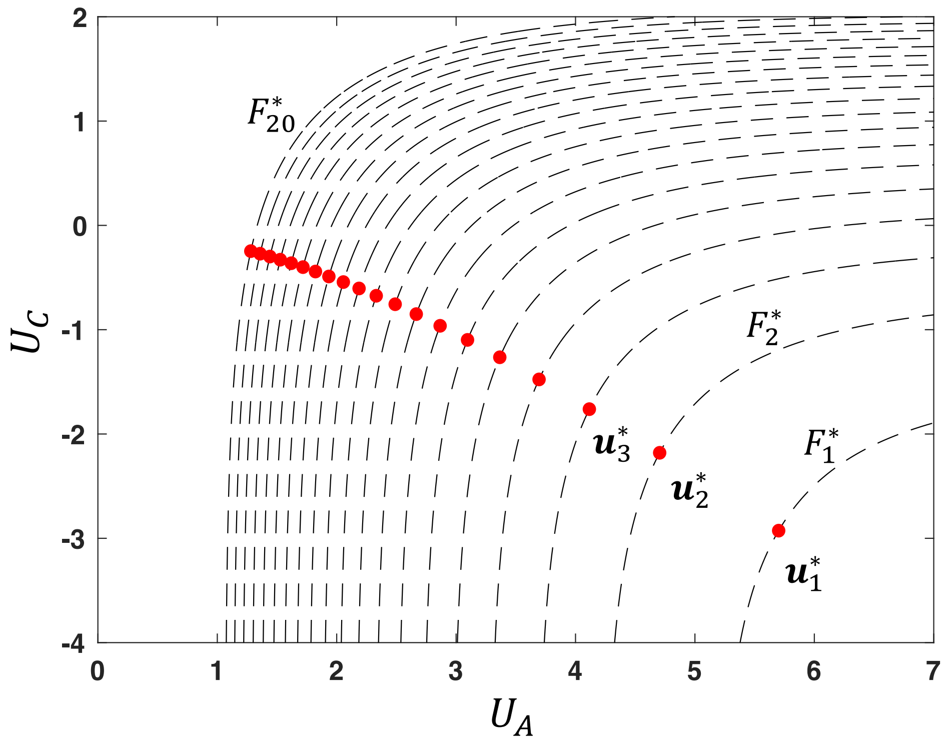
Reference: Au S.K. and Beck J.L. Estimation of small failure probabilities in high dimensions by subset simulation. Probabilistic Engineering Mechanics, Volume 16, 2001, pp. 263-277.

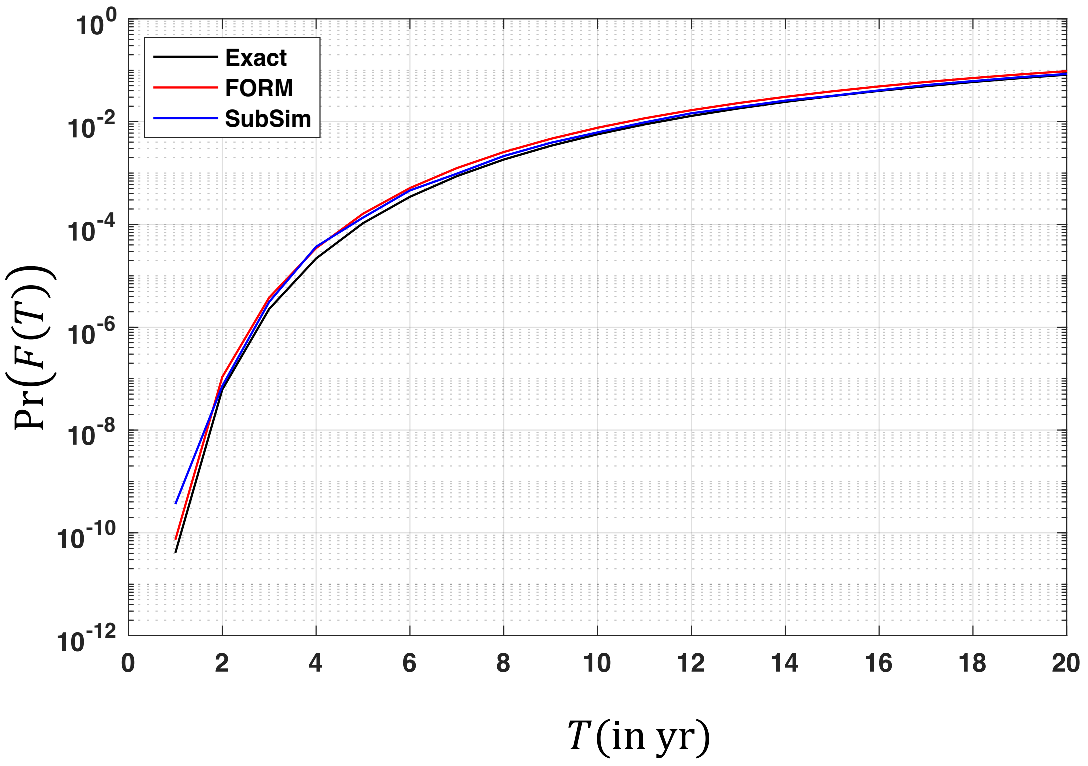
Example 1 - with FORM and subset simulation

Limit state function in the standard normal space:

$G_j(\mathbf{U}) = w - e^{(\mu_{\ln A}+\sigma_{\ln A}U_A)}\left(t_j - e^{(\mu_{\ln C}+\sigma_{\ln C}U_C)}\right)$, where $\mathbf{U} = \{U_A, U_C\}$.

Consider $t_1 = 1\text{yr}$, $t_2 = 2\text{yr},\ldots,t_{20} = 20\text{yr}$. In this example, $F_j^* = F^*(t_j) = F(t_j)$.





Example 1 - sequential subset simulation

For $t_1 < t_2 \ldots < t_{20}$, we have $F_{20}^* \supseteq F_{19}^* \supseteq \cdots \supseteq F_1^*$.

The sequence of failure probabilities $\Pr(F_1^*), \ldots, \Pr(F_{20}^*)$ can be estimated through sequential subset simulation approach.

$$\Pr\left(F_j^*\right) = \Pr(F_{20}^*) \prod_{k=0}^{20-j-1} \Pr(F_{20-k-1}^* | F_{20-k}^*)$$

- The failure probability $\Pr(F_{20}^*)$ is estimated using the standard subset simulation approach.
- If $\Pr(F_{20-k-1}^* | F_{20-k}^*) \leq 0.1$, additional subset levels can be introduced between F_{20-k}^* and F_{20-k-1}^*, as in standard subset simulation.
- If $\Pr(F_{20-k-1}^* | F_{20-k}^*) \geq 0.1$, use the failure samples in F_{20-k}^* to estimate $\Pr(F_{20-k-1}^* | F_{20-k}^*)$, $\Pr(F_{20-k-2}^* | F_{20-k}^*)$, $\Pr(F_{20-k-3}^* | F_{20-k}^*), \ldots$, $\Pr(F_{20-k-r}^* | F_{20-k}^*)$, until $\Pr(F_{20-k-r-1}^* | F_{20-k}^*) \leq 0.1$.
- Generate failure samples in F_{20-k-r}^* and continue the procedure.

Reference: Straub D., Schneider R., Bismut E. and Kim H.Y. Reliability analysis of deteriorating structural systems. Structural Safety, Volume 82, 2020, Article 101877.

Example 2

The capacity of a system is described by a linearized deterioration model:

$$R(t) = r_0 - At \ ,$$

the initial resistance $r_0 = 49.5\text{MPa}$ is deterministic and A is a normal random variable with mean $\mu_A = 0.2\text{MPa/yr}$ and standard deviation $\sigma_A = 0.2\text{MPa/yr}$.

The load effect is described by its annual maximum, $S_{max,j}$, which is i.i.d. for all years. $S_{max,j}$ is a normal random variable with mean $\mu_S = 40\text{MPa}$ and standard deviation $\sigma_S = 2\text{MPa}$.

$F(T) = \bigcup_{j=1}^{m} F_j^*$ where $F_j^* = \left\{\min_{\tau \in (t_{j-1}, t_j]} R(\tau) - S(\tau) \leq 0\right\}$, for yearly intervals $m = T$ and $t_1 = 1\text{yr}$, $t_2 = 2\text{yr}, \ldots$

Without deterioration:

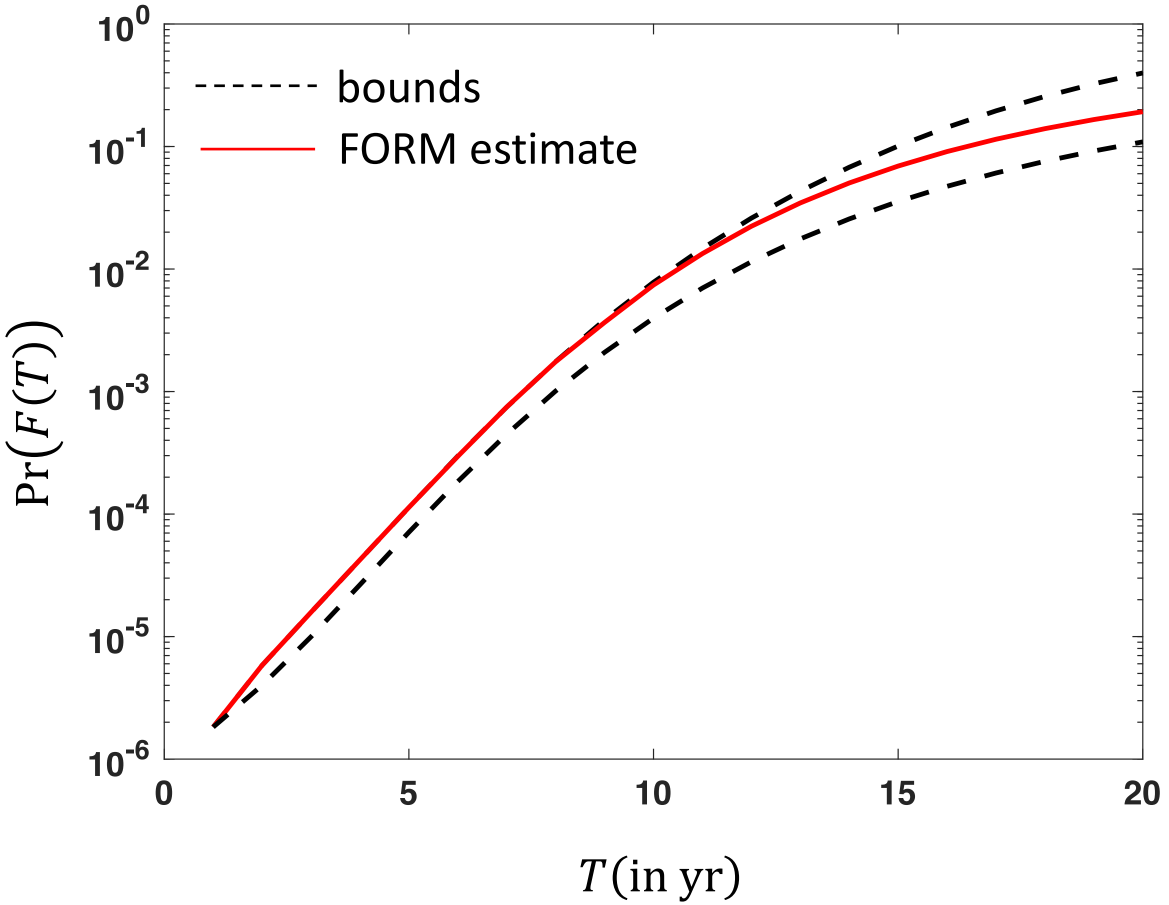
- Interval failure probability: $\Pr\left(F_j^*\right) = \Pr\left\{S_{max,j} > r_0\right\}$
- $\Pr\left(F(T)\right) = 1 - \prod_{j=1}^{m} \Pr\left(F_j^{*c}\right) = 1 - \left(F_{S_{max,j}}(r_0)\right)^m = 2 \cdot 10^{-5}$ for $T = 20\text{yr}$.

Example 2 (2)

With deterioration:

- Interval failure probability: $\Pr(F_j^*) \approx \Pr\{S_{max,j} > R(t_j)\} = \Pr\{g_j(\mathbf{Y}) \leq 0\}$, where $\mathbf{Y} = \{A, S_{max,j}\}$ and $g_j(\mathbf{Y}) = r_0 - At_j - S_{max,j}$.
- Limit state in the standard normal space: $G_j(\mathbf{U}) = (r_0 - \mu_A t_j - \mu_S) - t_j\sigma_A U_A - \sigma_S U_{S,j}$.
- Based on FORM:
    - Design point $\boldsymbol{u}_j^* = \left(\frac{t_j\sigma_A(r_0-\mu_A t_j-\mu_S)}{t_j^2\sigma_A^2+\sigma_S^2}, \frac{\sigma_S(r_0-\mu_A t_j-\mu_S)}{t_j^2\sigma_A^2+\sigma_S^2}\right)$.
    - $\beta_j = \|\boldsymbol{u}_j^*\| = \frac{r_0-\mu_A t_j-\mu_S}{\sqrt{t_j^2\sigma_A^2+\sigma_S^2}}$, $\boldsymbol{\alpha}_j = \left(\frac{t_j\sigma_A}{\sqrt{t_j^2\sigma_A^2+\sigma_S^2}}, \frac{\sigma_S}{\sqrt{t_j^2\sigma_A^2+\sigma_S^2}}\right)$.
    - $\Pr(F_j^*) \approx \Pr\{G_j(\mathbf{U}) \leq 0\} = \Phi(-\beta_j)$.
    - $\Pr(F(T)) \approx 1 - \Phi_m(\boldsymbol{\beta}; \boldsymbol{\rho})$ where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)$ and $[\boldsymbol{\rho}]_{j,k} = \alpha_{A,j}\alpha_{A,k} = \frac{t_j t_k \sigma_A^2}{\sqrt{t_j^2\sigma_A^2+\sigma_S^2}\sqrt{t_k^2\sigma_A^2+\sigma_S^2}}$

Example 2 (3)



FORM series system reliability bounds: $\max_{j=1,\ldots,m}\Pr(F_j^*) \le \Pr(F(T)) \le 1 - \prod_{j=1}^{m}\left(1 - \Pr(F_j^*)\right)$

Outcrossing theory-based approach

Consider the failure event:

$$F(T) = \{\exists t \in [0, T]: \min_{0 \le t \le T} g(\mathbf{Y}, t) \le 0\}$$

where $g(\mathbf{Y}, t) = 0$ is the limit state. $\mathbf{Y}$ is comprised of all random variables and random processes in the structural reliability problem.

In the general case, $g(\mathbf{Y}, t)$ is a random process. Let $N(t)$ denote the number of out-crossings of $g(\mathbf{Y}, t)$ from the safe state to the unsafe safe in the time duration $(0, t]$.

The probability of failure for the time duration $[0, T]$ can be evaluated based on out-crossing theory:

$$P_F(0, T) = \Pr(\{g_0 \le 0\} \cup \{N(T) > 0\}) = 1 - (1 - \Pr\{g_0 \le 0\})\Pr\{N(T) = 0 | g_0 \ge 0\}$$

g_0 is the limit state function value at $t = 0$. Failure in the time interval $[0, T]$ corresponds either to failure at $t = 0$ or to a later outcrossing of the limit state surface if the system is in safe state at $t = 0$.

The Poisson approximation

- For a regular process $g(\mathbf{Y}, t)$, the outcrossing rate $\nu(t)$ is defined as:

$$\nu(t) = \lim_{\Delta t \to 0, \Delta t > 0} \frac{\Pr\{N(t+\Delta t) - N(t) = 1\}}{\Delta t} = \lim_{\Delta t \to 0, \Delta t > 0} \frac{\Pr(\{g_t > 0\} \cap \{g_{t+\Delta t} \leq 0\})}{\Delta t}$$

- The expected number of outcrossings in the time interval $(0, t]$ is given by $\mathrm{E}[N(t)] = \int_0^t \nu(\tau) d\tau$.
- If the occurrence of outcrossings is modeled by a Poisson point process
  - the probability of no outcrossing in time interval $(0, \mathrm{T}]$ is approximated as:

$$\Pr\{N(T) = 0 | g_0 \geq 0\} = \frac{\left[\int_0^T \nu(t) dt\right]^0 e^{-\int_0^T \nu(t) dt}}{0!} = e^{-\int_0^T \nu(t) dt}$$

  - the failure probability for the time interval $[0, T]$ is approximately evaluated as:

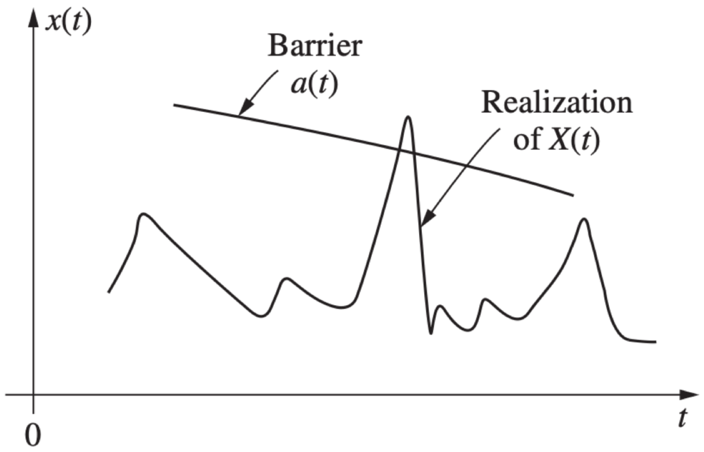
$$P_F(0, T) = 1 - (1 - \Pr\{g_0 \leq 0\}) e^{-\int_0^T \nu(t) dt}$$

- The probability $\Pr\{g_0 \leq 0\}$ can be evaluated using time-invariant reliability methods.

Outcrossing rate: Rice's formula

We consider $g(\mathbf{Y}, t) = a(t) - G(\mathbf{Y}, t)$, i.e., failure occurs when the process $G(\mathbf{Y}, t)$ outcrosses the barrier $a(t)$.

Define $X(t) \equiv G(\mathbf{Y}, t)$. It is assumed $X(t)$ is differentiable.



- Consider a segment of the sample $x(t)$ between the time instants t_1 and $t_1 + dt$.
- For sufficiently small dt, the curves can be taken as straight lines:
  - $x(t_1 + dt) = x(t_1) + \dot{x}dt$
  - $a(t_1 + dt) = a(t_1) + \dot{a}dt$
- Outcrossing occurs at time $t_1 + dt$:
  - $x(t_1) \leq a(t_1)$
  - $x(t_1 + dt) \geq a(t_1 + dt)$, i.e., $\dot{x}dt - \dot{a}dt \geq a(t_1) - x(t_1)$

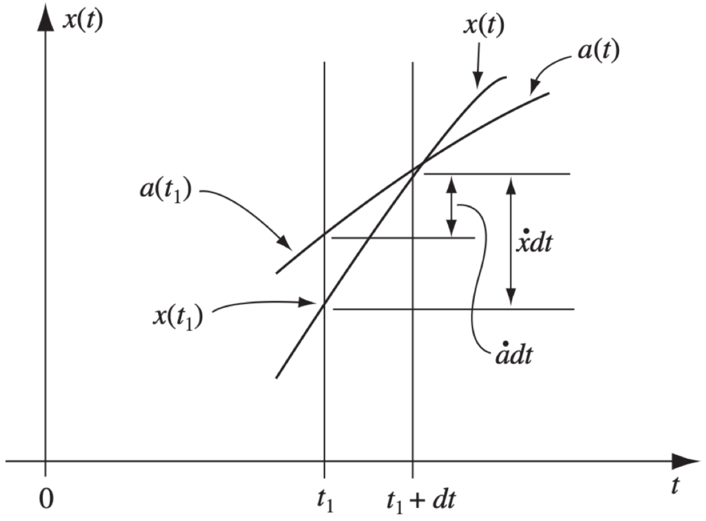


Image Source: Melchers R.E. and Beck A.T. Structural reliability analysis and prediction. 2018 John Wiley and Sons Ltd.

Outcrossing rate: Rice's formula

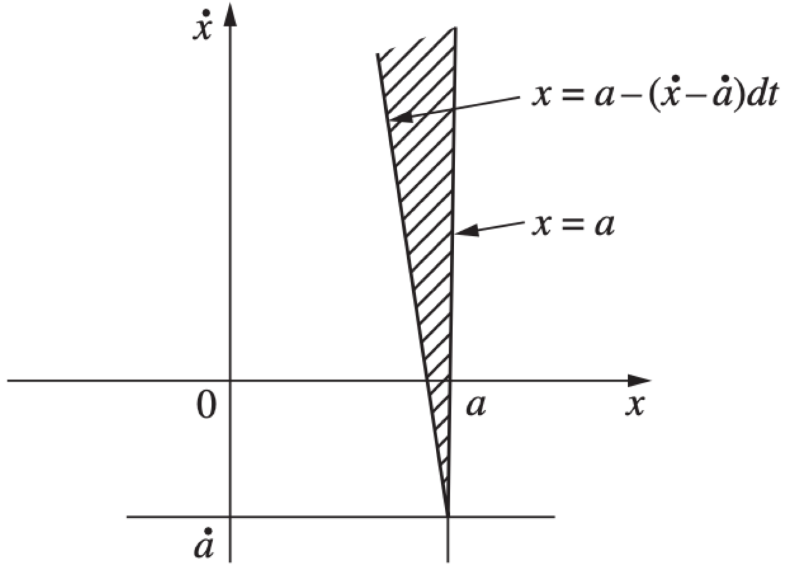
- Number of out-crossings in the time interval dt:

$$N = \int_{\dot{a}}^{\infty} \int_{a-(\dot{a}-\dot{x})dt}^{a} f_{X\dot{X}}(x, \dot{x}; t_1) dx d\dot{x}$$

$f_{X\dot{X}}(x, \dot{x}; t_1)$ is the joint PDF of $X(t)$ and $\dot{X}(t)$ at $t = t_1$

- As $dt \to 0$, $f_{X\dot{X}}(x, \dot{x}; t_1)$ can be approximated by $f_{X\dot{X}}(a, \dot{x}; t_1)$ over the range $a - (\dot{a} - \dot{x})dt \leq x \leq a$:

$$N = \int_{\dot{a}}^{\infty} (\dot{a} - \dot{x}) dt f_{X\dot{X}}(a, \dot{x}; t_1) d\dot{x}$$



- Outcrossing rate at time $t = t_1$:

$$\nu(t_1) = \lim_{dt \to 0} \frac{N}{dt} = \int_{\dot{a}}^{\infty} (\dot{a} - \dot{x}) f_{X\dot{X}}(a, \dot{x}; t_1) d\dot{x}$$

  - If a is time-independent, $\dot{a} = 0$.
  - Knowledge of $f_{X\dot{X}}(a, \dot{x}; t_1)$ for all $t_1 \in (0, T]$ is required.

Special case of a Gaussian process

- If $X(t)$ is a stationary Gaussian process:
  - $\mu_X(t) = \mathrm{E}[X(t)] \equiv \mu_X$, $\sigma_X^2(t) = E\left[\left(X(t) - \mu_X(t)\right)^2\right] \equiv \sigma_X^2$, $\mu_{\dot{X}}(t) = 0$ and $\sigma_{\dot{X}}^2(t) \equiv \sigma_{\dot{X}}^2$.
  - The random variables $X(t)$ and $\dot{X}(t)$ are independent:

$$f_{X\dot{X}}(a(t_1), \dot{x}; t_1) = \frac{1}{2\pi\sigma_X\sigma_{\dot{X}}} \exp\left\{-\frac{1}{2}\left[\left(\frac{a(t_1)-\mu_X}{\sigma_X}\right)^2 + \frac{\dot{x}^2}{\sigma_{\dot{X}}^2}\right]\right\}$$

  - If $a(t) \equiv a$, the outcrossing rate $\nu(t) = \frac{\sigma_{\dot{X}}}{2\pi\sigma_X} \exp\left\{-\frac{1}{2}\left(\frac{a-\mu_X}{\sigma_X}\right)^2\right\}$, for all $t \in (0, T]$.

- If $X(t)$ is a non-stationary Gaussian process:

$$\nu(t) = \frac{\sigma_{\dot{X}}(t)\sqrt{1-\rho_{X\dot{X}}^2(t)}}{\sigma_X(t)} \phi\left(\frac{a(t)-\mu_X(t)}{\sigma_X(t)}\right)\left\{\phi\left(-h(t)\right) + h(t)\Phi\left(h(t)\right)\right\}$$

where $h(t) = \frac{\rho_{X\dot{X}}(t)\left(a(t)-\mu_X(t)\right)}{\sigma_X(t)\sqrt{1-\rho_{X\dot{X}}^2(t)}}$ and $\rho_{X\dot{X}}(t)$ is the correlation coefficient between $X(t)$ and $\dot{X}(t)$.

Example 3 - Linear dynamic system

- Idealized SDOF structure:

$$m\ddot{X}(t) + c\dot{X}(t) + kX(t) = P(t)$$

Initial conditions: $X(0) = 0$, $\dot{X}(0) = 0$

- Failure event:

$$F(T) = \{\min_{0\leq t\leq T} x^* - X(t) \leq 0\}$$

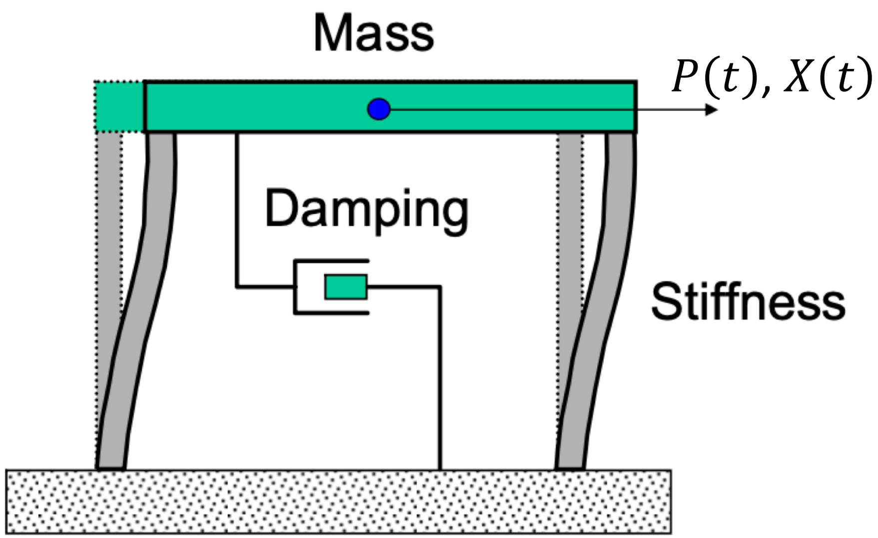
- Displacement $X(t)$ (assuming system is time-invariant):

$$X(t) = \int_0^t h_X(t-\tau)P(\tau)d\tau$$

$h_X(t)$ is the impulse response function, given by:

$$h_X(t) = \frac{1}{m\omega_d} e^{\eta\omega t} \sin(\omega_d t);\ t \geq 0$$

with $\omega = \sqrt{\frac{k}{m}}$, $\omega_d = \omega\sqrt{1-\eta^2}$, $\eta = \frac{c}{2m\omega}$.



$P(t)$ is a Gaussian random process

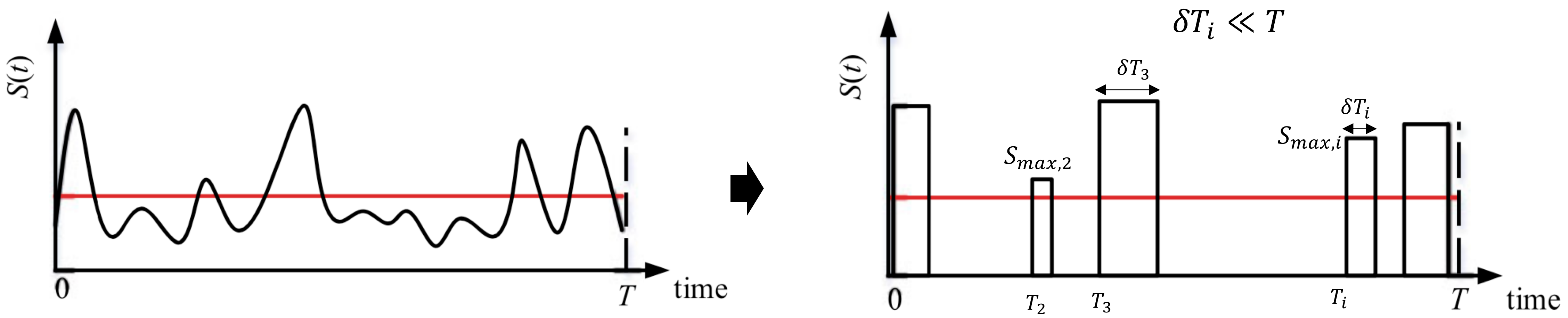
Example 3 (2)

- We consider $P(t)$ is a Gaussian random process
  - $X(t)$ is a Gaussian random process
  - $\dot{X}(t) = \frac{d}{dt}X(t) = \int_0^t h_{\dot{X}}(t-\tau)P(\tau)d\tau$ is a Gaussian random process, where $h_{\dot{X}}(t) = \frac{d}{dt}h_X(t)$.

- Joint statistics of $X(t)$ and $\dot{X}(t)$:
  - $\mu_X(t) = \mathrm{E}[X(t)] = \int_0^t h_X(t-\tau)\mu_P(\tau)d\tau$, where $\mu_P(t) = \mathrm{E}[P(t)]$.
  - $\mu_{\dot{X}}(t) = \mathrm{E}[\dot{X}(t)] = \int_0^t h_{\dot{X}}(t-\tau)\mu_P(\tau)d\tau$.
  - $\sigma_X^2(t) = \mathrm{E}\left[\left(X(t) - \mu_X(t)\right)^2\right] = \int_0^t \int_0^t h_X(t-\tau_1)K_{PP}(\tau_1,\tau_2)h_X(t-\tau_2)d\tau_1 d\tau_2$.
  - $\sigma_{\dot{X}}^2(t) = \int_0^t \int_0^t h_{\dot{X}}(t-\tau_1)K_{PP}(\tau_1,\tau_2)h_{\dot{X}}(t-\tau_2)d\tau_1 d\tau_2$
  - $\mathrm{COV}\left(X(t),\dot{X}(t)\right) = \mathrm{E}\left[\left(X(t)-\mu_X(t)\right)\left(\dot{X}(t)-\mu_{\dot{X}}(t)\right)\right] = \int_0^t \int_0^t h_X(t-\tau_1)K_{PP}(\tau_1,\tau_2)h_{\dot{X}}(t-\tau_2)d\tau_1 d\tau_2$

  where $K_{PP}(\tau_1,\tau_2) = \mathrm{E}\left[\left(P(\tau_1)-\mu_P(\tau_1)\right)\left(P(\tau_2)-\mu_P(\tau_2)\right)\right]$.

Discretized approach: random number of discrete events



- The discretization is in terms of the number of occurrence of a particular event, e.g., a storm or an earthquake of a particular duration.
- The number of discrete events as well as the time of occurrence is random.
- The interval failure events: $F_j^*(\mathbf{T}) = \left\{\min_{\tau\in(T_j, T_j+\delta T_j]} R(\tau) - S(\tau) \leq 0\right\}$, where $\mathbf{T} = \{T_1, \dots, T_k\}$ is the random vector comprising of the occurrence time instants.

Load occurrence model

- Load occurrence is commonly modelled by a Poisson point process.

- For a stationary Poisson process with occurrence rate λ:
  - PMF of the number of loads in $[0, T]$ is $\Pr(N(T) = k) = \frac{(\lambda T)^k e^{-\lambda T}}{k!}$; $k = 0,1,2, \ldots$.
  - Joint PDF of the occurrence times is given by $f_{\mathbf{T}}(\mathbf{t}) = \left(\frac{1}{T}\right)^k$, where $\mathbf{t} = \{t_1, \ldots, t_k\} \in [0, T]^k$.

- For a non-stationary Poisson process with time-variant occurrence rate $\lambda(t)$:
  - PMF of the number of loads in $[0, T]$ is $\Pr(N(T) = k) = \frac{\left(\int_0^T \lambda(t)dt\right)^k e^{-\int_0^T \lambda(t)dt}}{k!}$; $k = 0,1,2, \ldots$.
  - Joint PDF of the occurrence times is given by $f_{\mathbf{T}}(\mathbf{t}) = \prod_{j=1}^{k} \frac{\lambda(t_j)}{\int_0^T \lambda(t)dt}$, where $\mathbf{t} = \{t_1, \ldots, t_k\} \in [0, T]^k$.

Solution approach

Consider the number of discrete events $N(T) = k$ and occurrence times $\mathbf{t} = (t_1, \dots, t_k) \sim f_{\mathbf{T}}(\mathbf{t})$.

The (conditional) time-dependent probability of failure:

$$\Pr\big(F(T|N(T) = k, \mathbf{T} = \mathbf{t})\big) = \Pr\big(\cup_{j=1}^{k} F_j^*(\mathbf{t})\big)$$

Considering the randomness in the load occurrence times, we get:

$$\Pr\big(F(T|N(T) = k)\big) = \int_0^T \cdots \int_0^T \Pr\big(\cup_{j=1}^{k} F_j^*(\mathbf{t})\big)\, f_{\mathbf{T}}(t_1, \dots, t_k) dt_1 \cdots dt_k$$

Taking into consideration the randomness associated with the number of load events:

$$\Pr\big(F(T)\big) = \sum_{k=0}^{\infty} \Pr(N(T) = k) \Pr\big(F(T|N(T) = k)\big)$$

Summary

- Basic approaches for time-variant reliability analysis: time-integrated approach, discretized approach and outcrossing theory-based approach.
- The first two methods enable the failure probability over the whole service life of the structure to be evaluated using time-invariant reliability methods.
- The outcrossing theory-based approach establishes the limit state function as a random process. The time-variant reliability is solved as a first-passage problem based on the Poisson approximation.
- The outcrossing rate of the limit state function can be evaluated using alternative methods: Rice's formula, PHI2 method etc.
- Methods based on Monte Carlo simulation are more versatile for time-variant reliability estimation.