# Bayesian techniques for updating reliability using data

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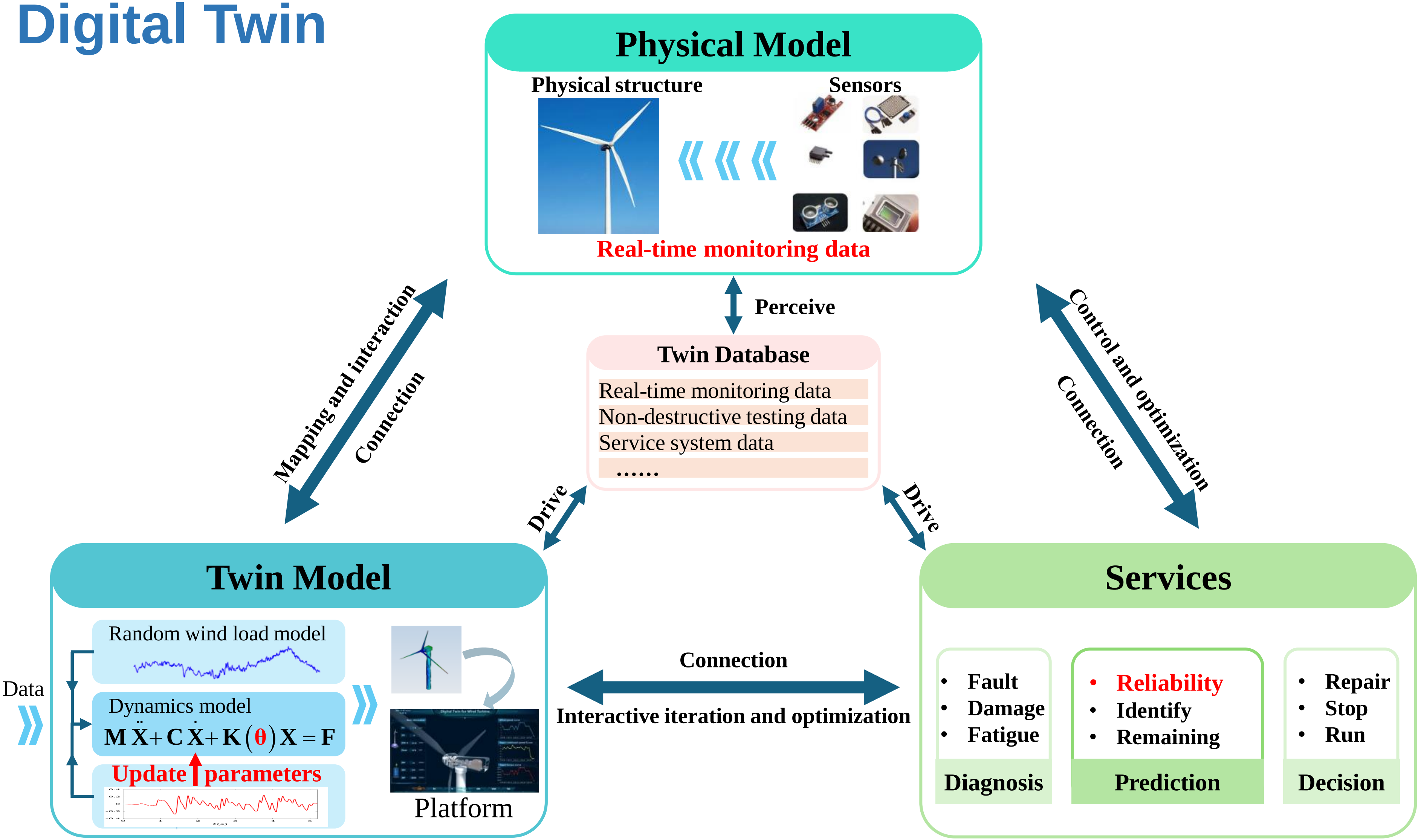
# Outline

# Background

## Digital Twin




**Virtual (Twin) models that replicate real-world systems, allowing for continuous monitoring, prediction, and updating based on monitoring data**

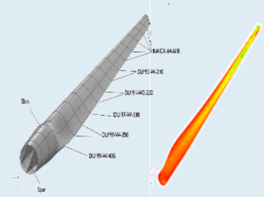
# Uncertainty

**Uncertainty in dynamical structure**

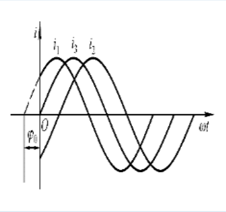
Epistemic imperfection

Can be reduced or eliminated

## Epistemic uncertainty



Model simplification



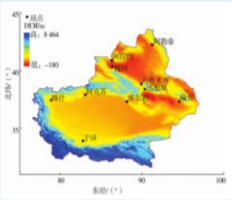
Data limitation

Sensor aging

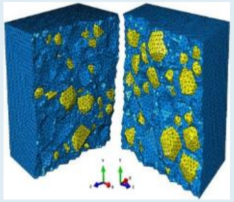
cannot be removed

Inherent uncertainty

## Aleatory uncertainty



Temperature uncertainty



Material uncertainty

Manufacture uncertainty

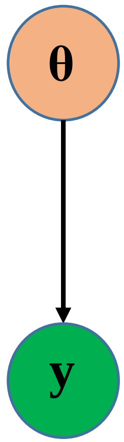
Assemble uncertainty

# Background

**1. Uncertainty plays an important role for model and reliability updating**

**2. Reasonably quantification of uncertainty is crucial for digital twining**

# Classical Bayesian Modeling(CBM)




**Dataset D**

- Bayes theorem

Likelihood — Prior PDF — Posterior PDF — Evidence

$$p(\boldsymbol{\theta}|\mathbf{D}) = \frac{p(\mathbf{D}|\boldsymbol{\theta})\,p(\boldsymbol{\theta})}{p(\mathbf{D})}$$

- Reliability analysis

$$P_F = \int_{\mathbf{x}\in\mathbb{R}^{N_\mathbf{x}}} \mathrm{I}_F(\mathbf{x})\,p(\mathbf{x})\,d\mathbf{x}$$

Failure event $F = \{\mathbf{x}\in\mathbb{R}^{N_\mathbf{x}} : g(\mathbf{x}) \le 0\}$

- Reliability updating
  - Prior probability of failure

$$P_F = \int_{\boldsymbol{\theta}\in\mathbb{R}^{N_\boldsymbol{\theta}}} \mathrm{I}_F(\boldsymbol{\theta})\,p(\boldsymbol{\theta})\,d\boldsymbol{\theta}$$

  - Posterior/updated probability of failure

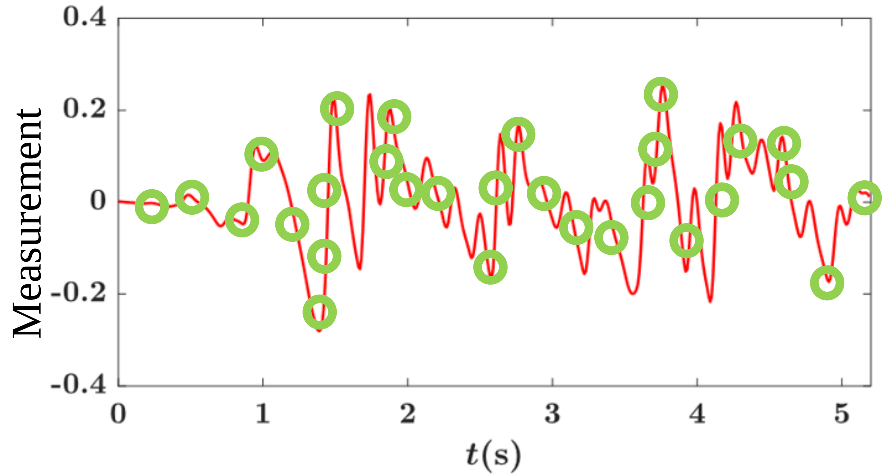
$$P_{F|\mathbf{D}} = \int_{\boldsymbol{\theta}\in\mathbb{R}^{N_\boldsymbol{\theta}}} \mathrm{I}_F(\boldsymbol{\theta})\,p(\boldsymbol{\theta}|\mathbf{D})\,d\boldsymbol{\theta}$$

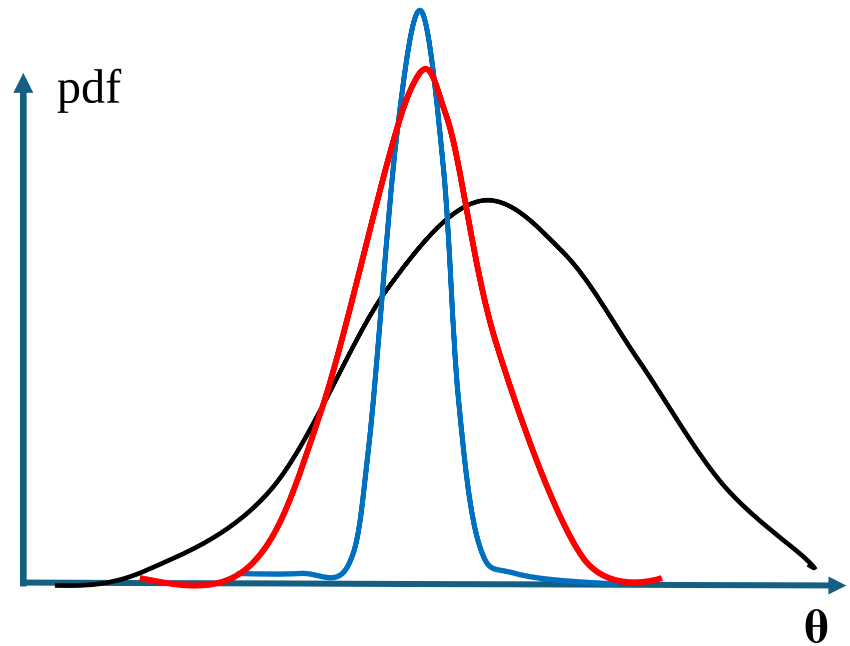
$$\boldsymbol{\theta} \sim p(\boldsymbol{\theta}|\mathbf{D})$$

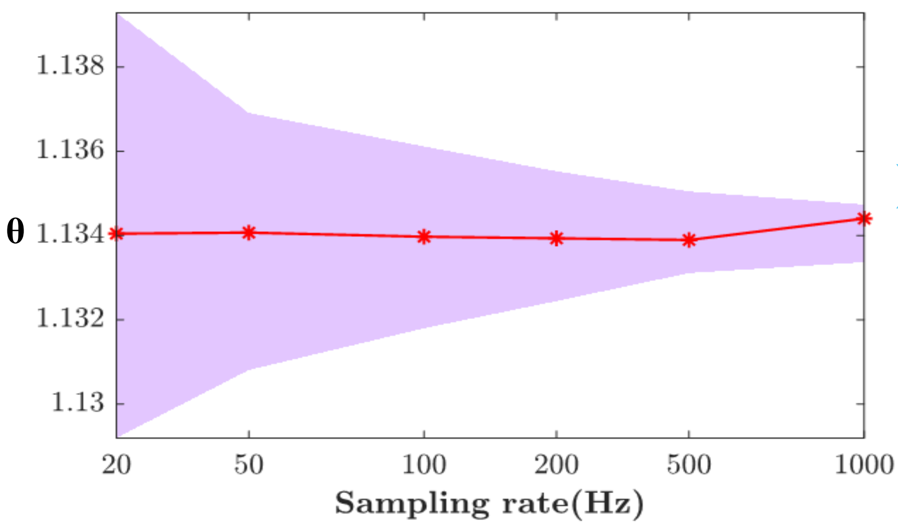
Papadimitriou, Beck, & Katafygiotis. (2001). Updating robust reliability using structural test data. *PEM*, *16*(2), 103-113.

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

Single dataset D




- Parameter uncertainty decreases as the number of data increases (Epistemic uncertainty)
- Reliability updates based on data

- ☐ **What if we have multiple datasets?**
- ☐ **Uncertainty due to variability (irreducible)?**

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

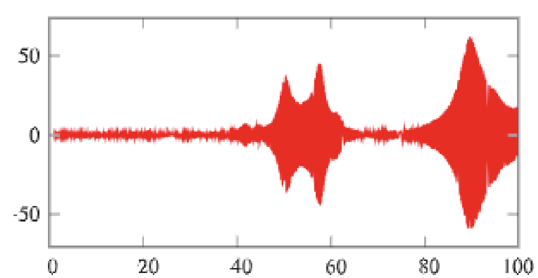
## Multiple datasets D

**Type I: many components/individuals/members in a population**

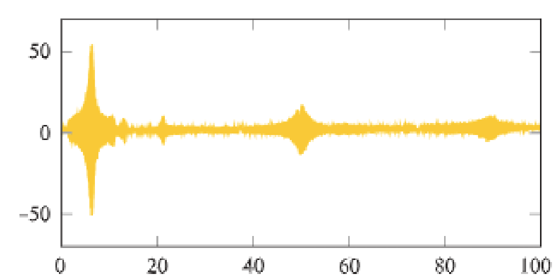
**Example: "Identical" components manufactured for the same car brand-model**

**Type II : multiple experiments in a specific system/component**

**Example: Single dynamical structure using many measurements (datasets)**




**Dataset $D_1$**




**Dataset $D_2$**

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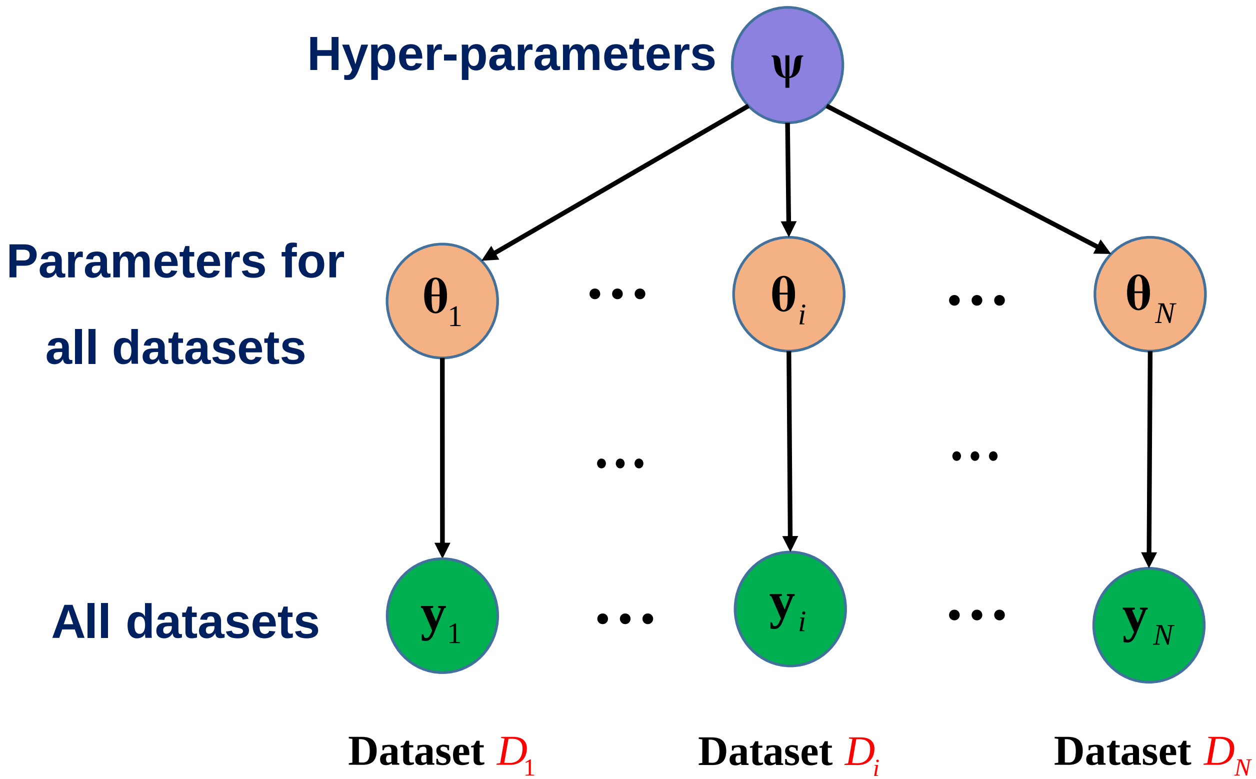
Ou, et al., SCHM, 28.6 (2021): e2660.

## Hierarchical Bayesian Modeling(HBM)




- Built on the existing Bayesian formulations
- Embed the uncertainty into model parameters
- Results in robust propagation of uncertainty and reliability updating

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

**Hyper-parameters** $\boldsymbol{\psi} = \{\boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta\}$

**Model parameters for all datasets** $\boldsymbol{\Theta} = \{\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_N\}$

**All datasets** $\boldsymbol{D} = \{D_1, \ldots, D_N\}$

Posterior distribution of all parameters → **Marginalization** → Posterior distribution of hyper parameters

**Bayes theorem**

- **Joint Posterior Distribution**

$$p(\boldsymbol{\Theta}, \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta \mid \boldsymbol{D}) \propto p(\boldsymbol{D} \mid \boldsymbol{\Theta}, \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta)\, p(\boldsymbol{\Theta} \mid \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta)\, p(\boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta)$$

- **Parameterized Prior distribution**

$$p(\boldsymbol{\Theta}, \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta) = p(\boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta) \prod_{i=1}^{N} p(\boldsymbol{\theta}_i \mid \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta)$$

- **Likelihood function**

$$p(\boldsymbol{D} \mid \boldsymbol{\Theta}, \boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta) = \prod_{i=1}^{N} p(D_i \mid \boldsymbol{\theta}_i)$$

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

Posterior distribution of all parameters → **Marginalization** → Posterior distribution of hyper parameters

## Marginalization

- **Posterior distribution of all parameters**

$$p(\{\boldsymbol{\theta}_i\}_{i=1}^{N}, \boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mid D) \propto p(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \prod_{i=1}^{N} N(\boldsymbol{\theta}_i \mid \boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \prod_{i=1}^{N} p(D_i \mid \boldsymbol{\theta}_i)$$

- **Asymptotic Approximation**

$$p(D_i \mid \boldsymbol{\theta}_i) \propto N(\boldsymbol{\theta}_i \mid \boldsymbol{\theta}_i^*, \boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*)$$

- **Marginal Posterior Distribution of Hyper-parameters**

$$p(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mid D) \propto p(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \prod_{i=1}^{N} N(\boldsymbol{\mu}_{\boldsymbol{\theta}} \mid \boldsymbol{\theta}_i^*, \boldsymbol{\Sigma}_{\boldsymbol{\theta}} + \boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*)$$

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

Posterior samples of hyper parameters → **Propagation** → Predictive distribution of model parameters and responses

## Predictive Distribution

- **Posterior Distribution of the Parameters of Structural Model.**

$$p(\boldsymbol{\theta} \mid D) = \int p(\boldsymbol{\theta} \mid \boldsymbol{\mu}_{\theta}^{(m)}, \boldsymbol{\Sigma}_{\theta}^{(m)})\, p(\boldsymbol{\mu}_{\theta}, \boldsymbol{\Sigma}_{\theta} \mid D)\, d\boldsymbol{\mu}_{\theta}\, d\boldsymbol{\Sigma}_{\theta}$$

$$\approx \frac{1}{N_s} \sum_{m=1}^{N_s} \mathrm{N}(\boldsymbol{\theta} \mid \boldsymbol{\mu}_{\theta}^{(m)}, \boldsymbol{\Sigma}_{\theta}^{(m)})$$

**Gaussian mixture**

- **Posterior Distribution of the structural responses.**

$$p(y \mid D) = \int p(y \mid \boldsymbol{\theta})\, p(\boldsymbol{\theta} \mid D)\, d\boldsymbol{\theta}$$

$$\approx \frac{1}{N_s} \sum_{m=1}^{N_s} p(y \mid \boldsymbol{\theta}^{(m)})$$

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

Posterior samples of hyper parameters → **Reliability** → Posterior probability of failure

## Reliability given Multiple Datasets

- **Posterior probability of failure given multiple datasets is**

$$P_{F|\mathbf{D}} = \int_{\boldsymbol{\theta}\in\mathbb{R}^{N_\theta}} \mathrm{I}_F(\boldsymbol{\theta})\, p(\boldsymbol{\theta}|\mathbf{D})\, d\boldsymbol{\theta} = \frac{1}{N_s}\sum_{m=1}^{N_s}\int_{\boldsymbol{\theta}\in\mathbb{R}^{N_\theta}} \mathrm{I}_F(\boldsymbol{\theta})\, \mathrm{N}(\boldsymbol{\theta}\,|\,\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})\, d\boldsymbol{\theta}$$

$$= \frac{1}{N_s}\sum_{m=1}^{N_s} F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})$$

Posterior samples of hyper parameters

**where** $F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})$ **is the failure probability conditional on the hyper sample** $\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}$ **, given by**

$$F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}) = \int_{\boldsymbol{\theta}\in\mathbb{R}^{N_\theta}} \mathrm{I}_F(\boldsymbol{\theta})\, \mathrm{N}(\boldsymbol{\theta}\,|\,\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})\, d\boldsymbol{\theta}$$

**Subset simulation could be implemented to compute** $F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})$

# Algorithm for Asymptotic HBM

**Step 1: Requires full model runs**

**Model inference for each data set**

**Estimate the most probable value**

**Evaluate the uncertainty**

**Repeat this step for all data sets**

**Step 2: Does not require full model runs**

**Generate samples of hyper parameters**

$$p(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mid D) \propto p(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) \prod_{i=1}^{N} N(\boldsymbol{\mu}_{\boldsymbol{\theta}} \mid \boldsymbol{\theta}_i^*, \boldsymbol{\Sigma}_{\boldsymbol{\theta}} + \boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*)$$

**Compute posterior uncertainty of output QoI and reliability**

# Algorithm for Full Sampling (FS) HBM

**Step 1: Requires full FE model runs**

**Model inference for each data set**

**obtain samples from the posterior $p(\theta_i | d_i, M_i)$ for each data set by using sampling**

**Need to run the models in this step**

**Also need to store evidence and likelihood value for each data set**

**Step 2: Does not require full FE model runs**

**Generate samples of hyper parameters**

$$p(\mathbf{d} | \boldsymbol{\psi}, M) \sim p(\boldsymbol{\psi} | M) \prod_{i=1}^{N} p(d_i | \boldsymbol{\psi}, M)$$

**where**

$$p(d_i | \boldsymbol{\psi}, M) \approx \frac{p(d_i | M_i)}{N_s} \sum_{k=1}^{N_s} \frac{p\left(\boldsymbol{\theta}_i^{(k)} | \boldsymbol{\psi}, M\right)}{p\left(\boldsymbol{\theta}_i^{(k)} | M_i\right)}$$

**Compute posterior uncertainty of output QoI and reliability**

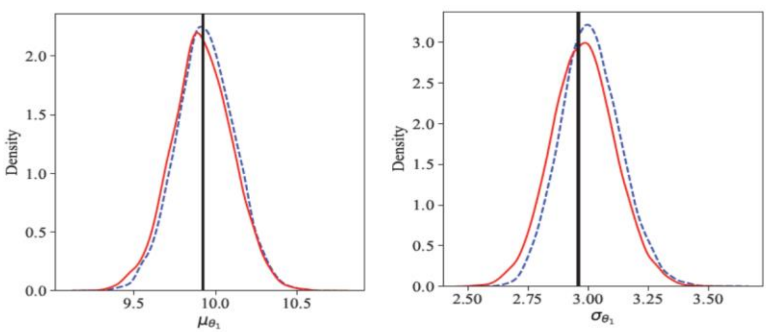
## ➢ Analytical solution for HBM

### Variational inference

Idea: to approximate the posterior distribution using variational distribution

Mean-field theory

$$q(\Xi) = q(\{\theta_i\}_{i=1}^N, \{\sigma_i^2\}_{i=1}^N) q(\boldsymbol{\psi}) q(\boldsymbol{\varphi})$$




To approximate

### HBM

The joint posterior distribution:

$$p(\Xi \mid D) \propto p(D \mid \Xi) p(\Xi)$$

Define parameter:

$$\Xi = \left\{ \{\theta_i\}_{i=1}^N, \{\sigma_i^2\}_{i=1}^N, \boldsymbol{\psi}, \boldsymbol{\varphi} \right\}$$

Model and hyper parameters:

Model parameter: $\left\{ \{\theta_i\}_{i=1}^N, \{\sigma_i^2\}_{i=1}^N \right\}$

Hyper parameter: $\left\{ \boldsymbol{\psi}, \boldsymbol{\varphi} \right\}$

Jia, et al. Mechanical Systems and Signal Processing, 2023, 189: 110060.

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

- **Factorized distribution**

$$q(\{\boldsymbol{\theta}_i\}_{i=1}^N, \{\sigma_i\}_{i=1}^N, \boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}, \mu_{\sigma^2}, \Sigma_{\sigma^2}) \propto q(\{\boldsymbol{\theta}_i\}_{i=1}^N, \{\sigma_i\}_{i=1}^N)\, q(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}})\, q(\mu_{\sigma^2}, \Sigma_{\sigma^2})$$

- **Marginal Posterior Distribution of Hyper-parameters**

Normal x Inverse Wishart

$$q(\boldsymbol{\mu}_{\boldsymbol{\theta}}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}) = \exp\left\{\mathrm{E}_{\Xi_{-\psi}}[\ln p(\Xi, D)]\right\} = \mathrm{N}\left(\boldsymbol{\mu}_{\boldsymbol{\theta}} \mid \tilde{\boldsymbol{\theta}}, \frac{1}{N}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}\right) \mathrm{IW}\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mid \Psi, \nu\right)$$

$$\tilde{\boldsymbol{\theta}} = \frac{1}{N}\sum_{i=1}^{N} \mathrm{E}_{\boldsymbol{\theta}_i}[\boldsymbol{\theta}_i] \quad \Rightarrow \quad \tilde{\boldsymbol{\theta}} \sim \frac{1}{N}\sum_{i=1}^{N} \boldsymbol{\theta}_i^*$$

$$\Psi = -N\tilde{\boldsymbol{\theta}}\tilde{\boldsymbol{\theta}}^T + \sum_{i=1}^{N} \mathrm{E}_{\boldsymbol{\theta}_i}[\boldsymbol{\theta}_i\boldsymbol{\theta}_i^T] \quad \Rightarrow \quad \Psi \sim \sum_{i=1}^{N} (\boldsymbol{\theta}_i^* - \tilde{\boldsymbol{\theta}})(\boldsymbol{\theta}_i^* - \tilde{\boldsymbol{\theta}})^T + \sum_{i=1}^{N} \boldsymbol{\Sigma}_{\boldsymbol{\theta}_i}^*$$

$$\nu = N - N_\theta - 2$$

**Generate Samples:** $\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)} \sim \mathrm{IW}\left(\boldsymbol{\Sigma}_{\boldsymbol{\theta}} \mid \Psi, \nu\right), \quad \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)} \sim \mathrm{N}\left(\boldsymbol{\mu}_{\boldsymbol{\theta}} \mid \tilde{\boldsymbol{\theta}}, \frac{1}{N}\boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}\right)$

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

## Analytical solution of hyper parameters

$$q(\boldsymbol{\mu}_\theta, \boldsymbol{\Sigma}_\theta) = \mathrm{N}\left(\boldsymbol{\mu}_\theta \mid \tilde{\boldsymbol{\theta}}, \frac{1}{N}\boldsymbol{\Sigma}_\theta\right) \mathrm{IW}\left(\boldsymbol{\Sigma}_\theta \mid \Psi, \nu\right)$$

**Total uncertainty**

$$\hat{\boldsymbol{\Sigma}}_\theta = \boldsymbol{\Sigma}_{AL} + \boldsymbol{\Sigma}_{\theta_i}$$

The summation of epistemic and aleatoric uncertainty

**Epistemic** uncertainty $\boldsymbol{\Sigma}_{\theta_i} = \frac{1}{N_d} J(\tilde{\boldsymbol{\theta}}) H^{-1}(\tilde{\boldsymbol{\theta}})$

- **It decreases with the increase of data points $N_d$.**

**Aleatoric** uncertainty $\boldsymbol{\Sigma}_{AL} = \frac{1}{N}\sum_{i=1}^{N}(E_{\theta_i}(\boldsymbol{\theta}_i) - \tilde{\boldsymbol{\theta}})(E_{\theta_i}(\boldsymbol{\theta}_i) - \tilde{\boldsymbol{\theta}})^T$

- **Aleatoric uncertainty does not decrease as the number of datasets $N$ increases.**

Hyper parameter uncertainty:

$$\boldsymbol{\Sigma}(\hat{\boldsymbol{\mu}}_\theta, \hat{\boldsymbol{\Sigma}}_\theta) = \frac{1}{N} H^{-1}(\hat{\boldsymbol{\mu}}_\theta, \hat{\boldsymbol{\Sigma}}_\theta)$$

- **$N$ is the number of data sets.**
- **Hyper parameter uncertainty decreases with the increase of data sets.**

Parameter mean value:

$$\hat{\boldsymbol{\mu}}_\theta = \tilde{\boldsymbol{\theta}} = \frac{1}{N}\sum_{i=1}^{N} E_{\theta_i}(\boldsymbol{\theta}_i)$$

- **The total parameter mean can be expressed as the mean value of solving parameters for each dataset**

# Bayesian Modeling Vs Hierarchical Bayesian Modeling

## Reliability given Multiple Datasets

- **Case 1: HBM – given sufficient number of datasets** ($\boldsymbol{\Sigma}(\hat{\boldsymbol{\mu}}_{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}) = \frac{1}{N} H^{-1}(\hat{\boldsymbol{\mu}}_{\boldsymbol{\theta}}, \hat{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}})$)
- **Case 2: HBM – considering all samples of hyper parameters**

**Case 1: Posterior probability of failure given multiple datasets:**

$$P_{F|\mathbf{D}} = \int_{\boldsymbol{\theta} \in \mathbb{R}^{N_{\boldsymbol{\theta}}}} \mathrm{I}_F(\boldsymbol{\theta}) \, p(\boldsymbol{\theta} | \mathbf{D}) \, d\boldsymbol{\theta} = \int_{\boldsymbol{\theta} \in \mathbb{R}^{N_{\boldsymbol{\theta}}}} \mathrm{I}_F(\boldsymbol{\theta}) \, \mathrm{N}(\boldsymbol{\theta} \,|\, \bar{\boldsymbol{\mu}}_{\boldsymbol{\theta}}, \bar{\boldsymbol{\Sigma}}_{\boldsymbol{\theta}}) d\boldsymbol{\theta}$$

**Case 2: Posterior probability of failure given multiple datasets:**

$$P_{F|\mathbf{D}} = \int_{\boldsymbol{\theta} \in \mathbb{R}^{N_{\boldsymbol{\theta}}}} \mathrm{I}_F(\boldsymbol{\theta}) \, p(\boldsymbol{\theta} | \mathbf{D}) \, d\boldsymbol{\theta} = \frac{1}{N_s} \sum_{m=1}^{N_s} \int_{\boldsymbol{\theta} \in \mathbb{R}^{N_{\boldsymbol{\theta}}}} \mathrm{I}_F(\boldsymbol{\theta}) \, \mathrm{N}(\boldsymbol{\theta} \,|\, \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}) d\boldsymbol{\theta}$$

$$= \frac{1}{N_s} \sum_{m=1}^{N_s} F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})$$

**where** $F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})$ **is the failure probability conditional on the sample** $\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}$ **, given by**

$$F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}) = \int_{\boldsymbol{\theta} \in \mathbb{R}^{N_{\boldsymbol{\theta}}}} \mathrm{I}_F(\boldsymbol{\theta}) \, \mathrm{N}(\boldsymbol{\theta} \,|\, \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}) d\boldsymbol{\theta}$$

**Subset simulation could be implemented to compute** $F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)})$

# Reliability given Multiple Datasets

**Special Case: Linear limit state function**

$$G(\boldsymbol{\theta}) = \boldsymbol{a}^T \boldsymbol{\theta} + \boldsymbol{b}$$

**The failure probability conditional on the sample $\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}$ is**

$$F^{(m)}(\boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}) = \int_{\boldsymbol{a}^T \boldsymbol{\theta} + \boldsymbol{b} \le 0} N(\boldsymbol{\theta} \mid \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)}, \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)}) d\boldsymbol{\theta} = \Phi\left(-\beta^{(m)}\right)$$

**where $\beta^{(m)}$ is the reliability index condition on a hyperparameter sample**

$$\beta^{(m)} = \frac{\boldsymbol{a}^T \boldsymbol{\mu}_{\boldsymbol{\theta}}^{(m)} + \boldsymbol{b}}{\boldsymbol{a}^T \boldsymbol{\Sigma}_{\boldsymbol{\theta}}^{(m)} \boldsymbol{a}^T}$$

**The posterior probability of failure given multiple datasets is**

$$\Pr(F) = \frac{1}{N_s} \sum_{m=1}^{N_s} \Phi\left(-\beta^{(m)}\right)$$

**the average of the failure probabilities over all hyperparameter samples.**

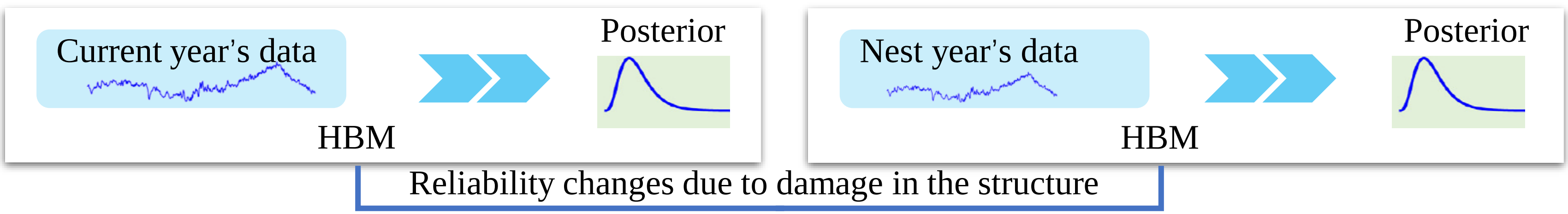
# Bayesian Modeling Vs Hierarchical Bayesian Modeling

## Two notes:

1. Apply HBM to datasets monitored periodically (e.g., year-by-year monitoring)

**Key Points**:

- Framework can be applied iteratively across monitorig periods.
- Variability due to damage introduces changes in reliability estimates.

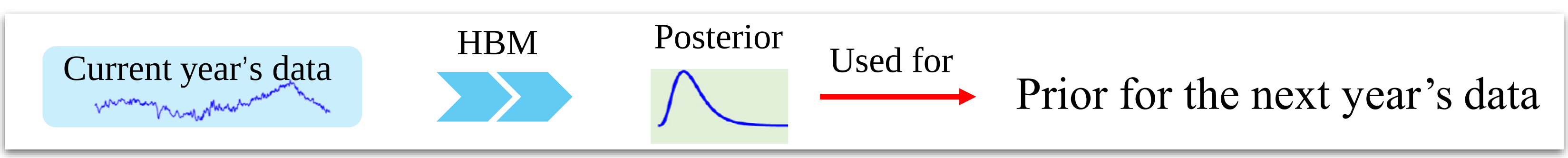



**2. Reliability Updates over Time**

Use the posterior distribution from the current year's analysis as the prior for the next year's data.

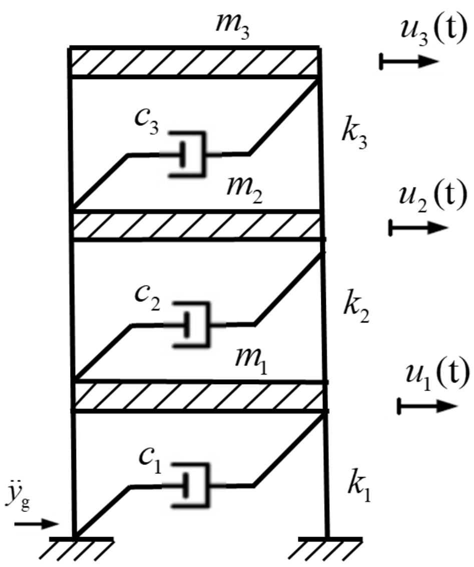
**Key Points**:

- HBM enables dynamic reliability updates.
- Ensures consistent integration of new information over time



# Application to Structural Dynamics

## Simulated example: 3-DOF linear structure with modal properties data

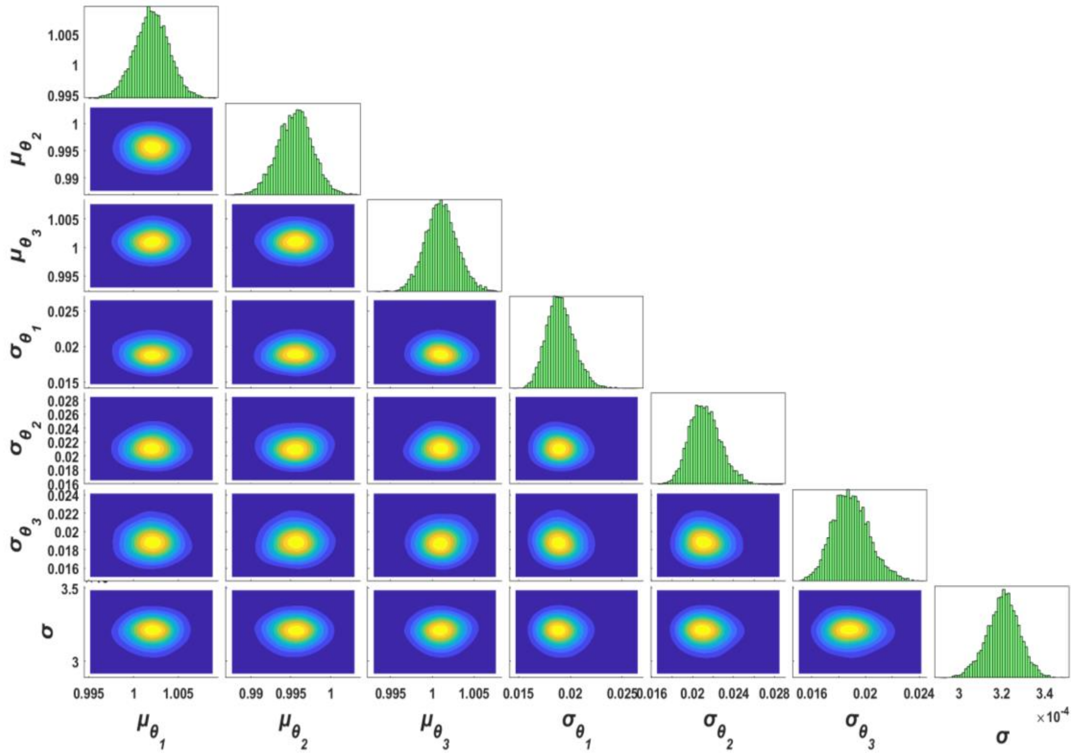



### Structural Model

1. Known Mass Matrix
2. Parameterized Stiffness $\mathbf{K} = \sum_{i=1}^{3} \theta_i \mathbf{K}_i$

### Probabilistic Model

1. Hyper-parameters
2. Measurements: 100 Sets of modal properties data

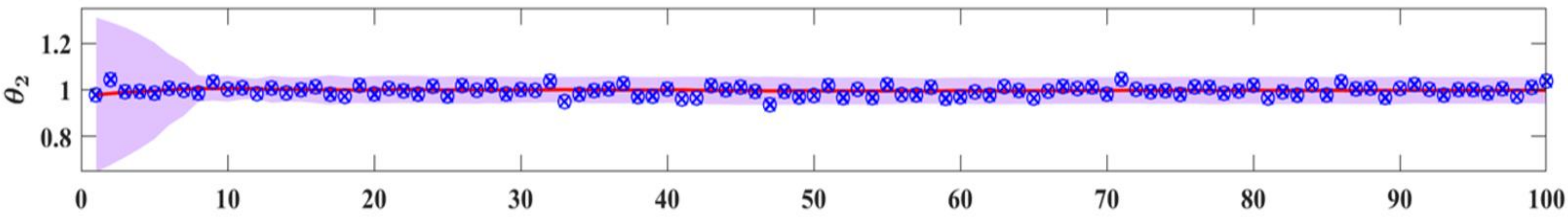


Posterior distribution of hyper parameters

# Application to Structural Dynamics

Mean values of hyper parameters using HBM (asymptotic, full sampling) and CBM

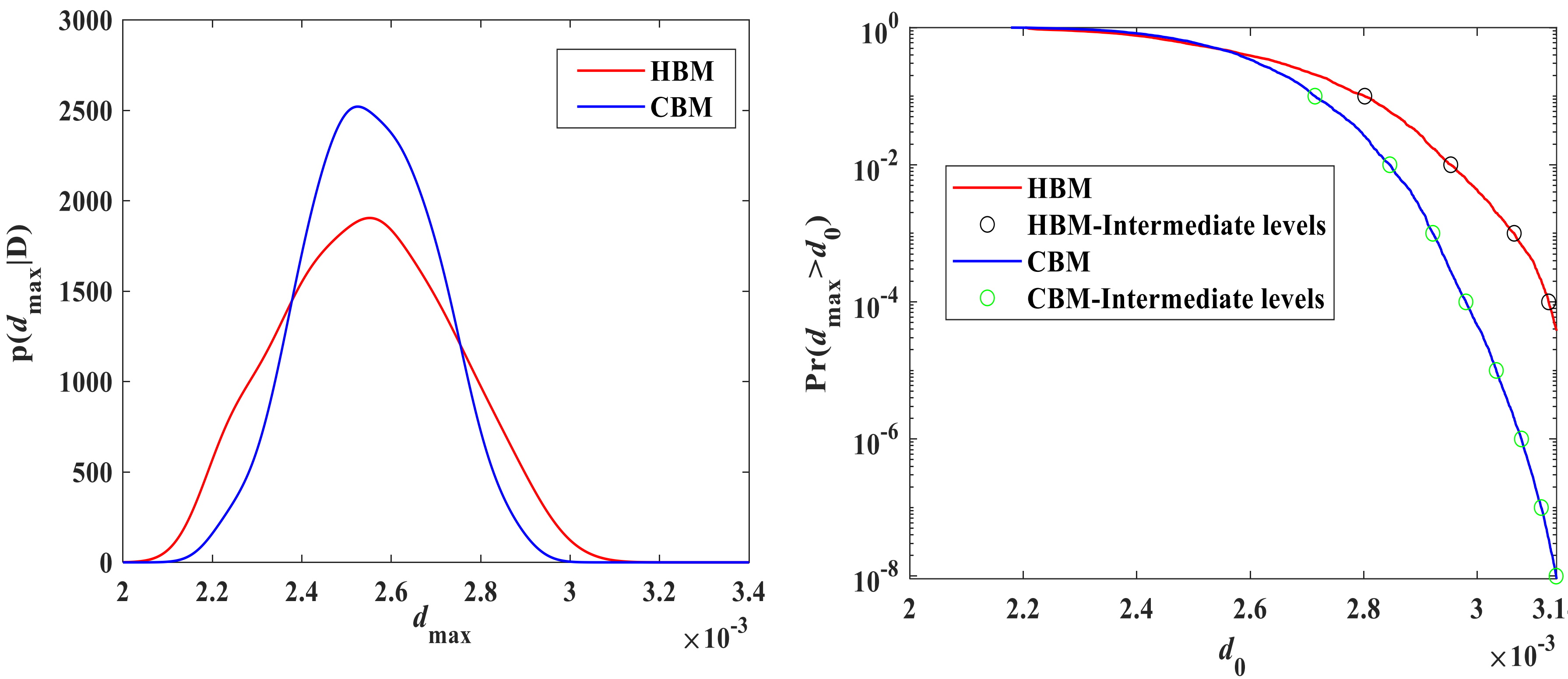
| Parameters | | $\hat{\mu}_{\theta_1}$ | $\hat{\sigma}_{\theta_1}$ | $\hat{\mu}_{\theta_2}$ | $\hat{\sigma}_{\theta_2}$ | $\hat{\mu}_{\theta_3}$ | $\hat{\sigma}_{\theta_3}$ |
|---|---|---|---|---|---|---|---|
| Methods | A-1(HBM) | 1.0024 | 0.0280 | 1.0008 | 0.0254 | 0.9947 | 0.0255 |
| | A-2(HBM) | 1.0007 | 0.0224 | 1.0004 | 0.0233 | 0.9955 | 0.0171 |
| | FS (HBM) | 1.0023 | 0.0217 | 1.0013 | 0.0234 | 0.9963 | 0.0177 |
| | CBM | 1.0023 | 0.0031 | 1.0011 | 0.0031 | 0.9947 | 0.0022 |




Identification uncertainty and parameter uncertainty as a function of the number of datasets $N_D$, with $N_D$ ranging from 1 to 100

# Application to Structural Dynamics

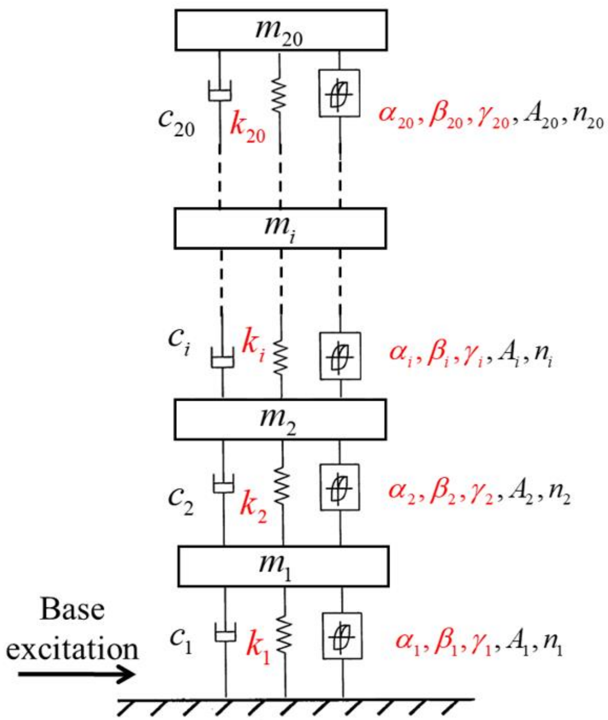
1000 parameters (Stochastic loads) + 3 model parameters

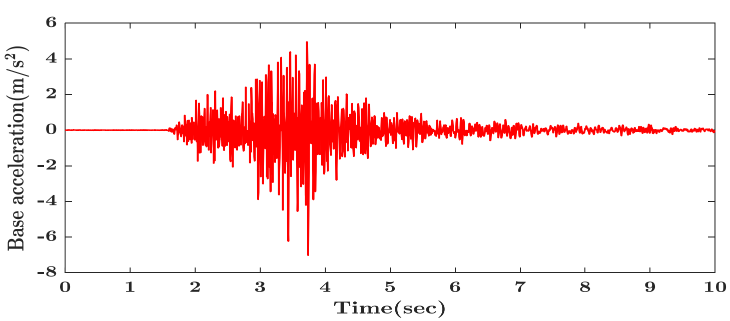


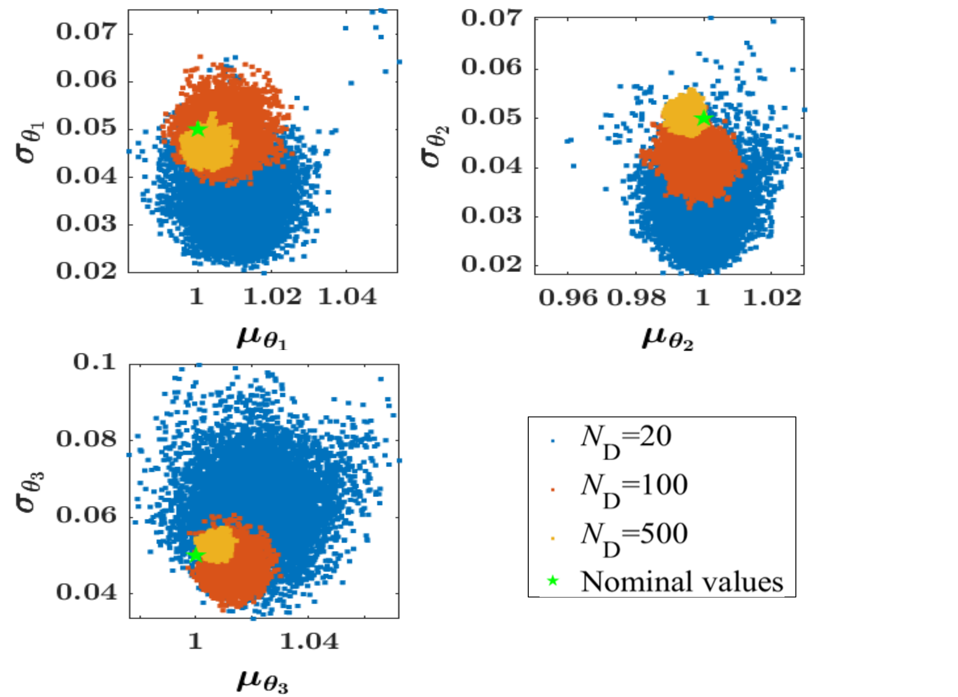

Posterior predictive distribution of the maximum drift in the simulated case and the exceedance probability of the maximum drift

# Application to Structural Dynamics

## Simulated example: 20-DOF nonlinear structure with time histories data



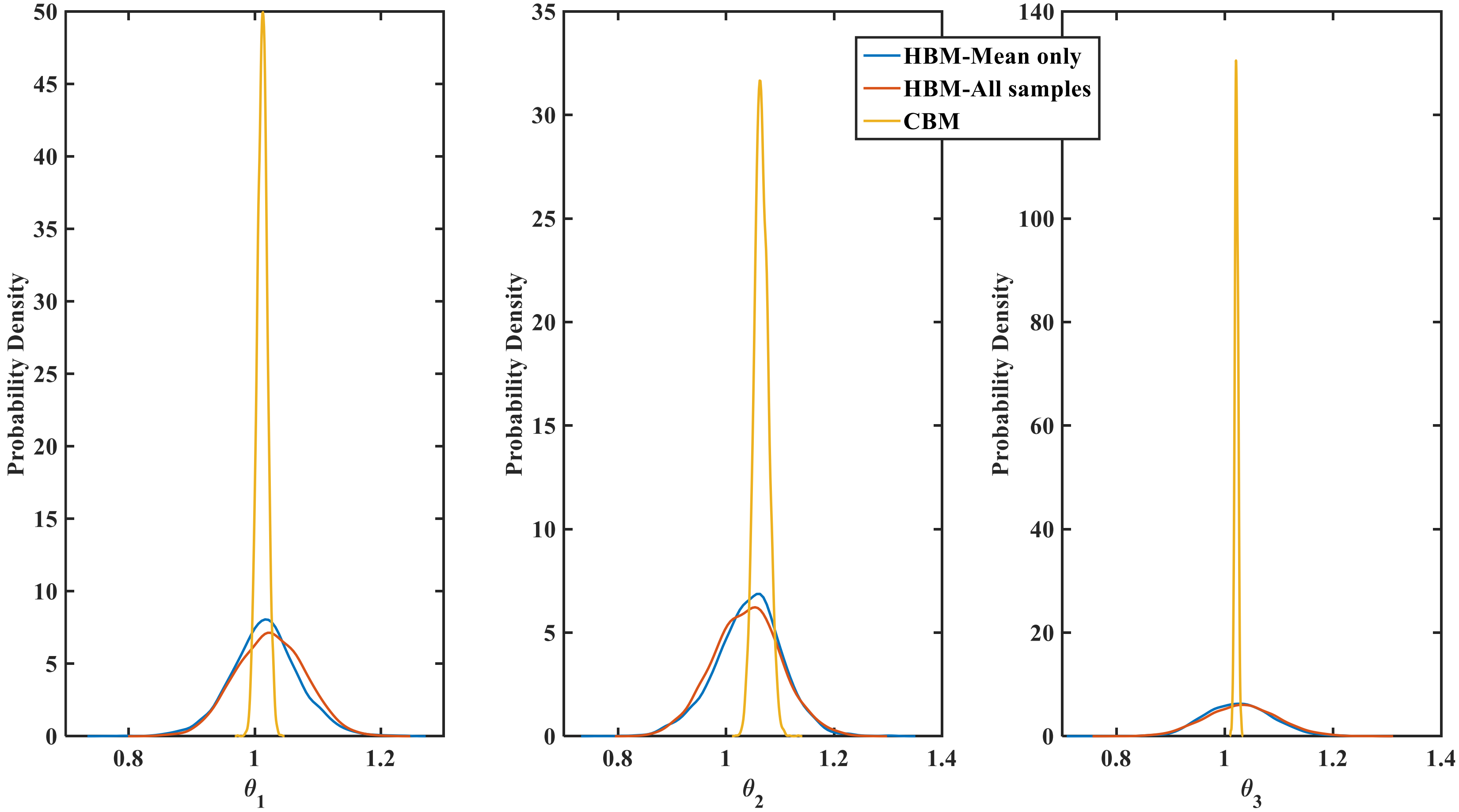




Posterior samples of hyper parameters using VI

- ***$N_D$* is the number of data sets.**
- **As the increase of the datasets, the uncertainty of the hyper parameters decreases.**

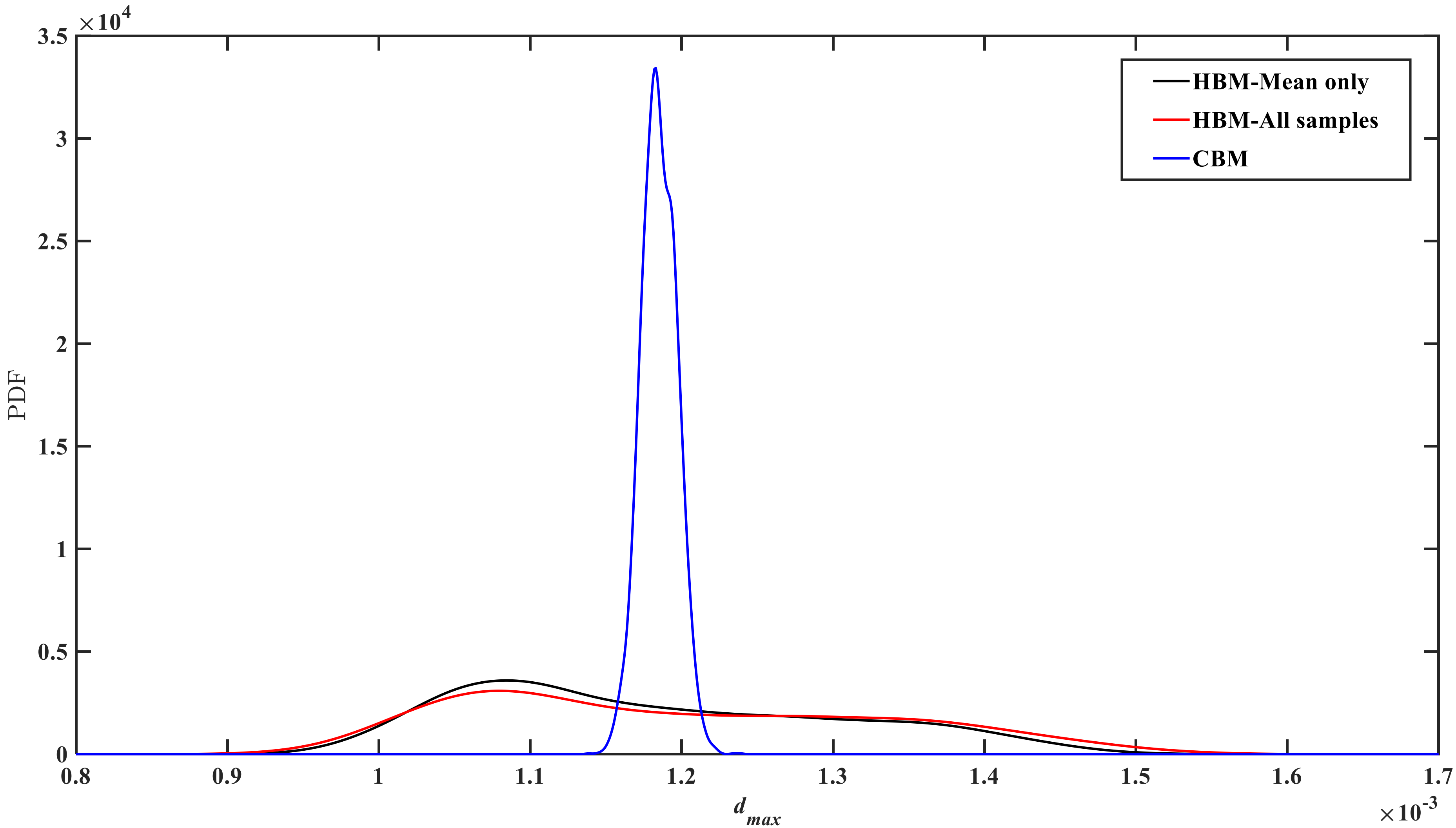
# Application to Structural Dynamics

- Case 1: HBM - only considering the means of hyper parameters
- Case 2: HBM - considering all samples of hyper parameters
- Case 3: CBM



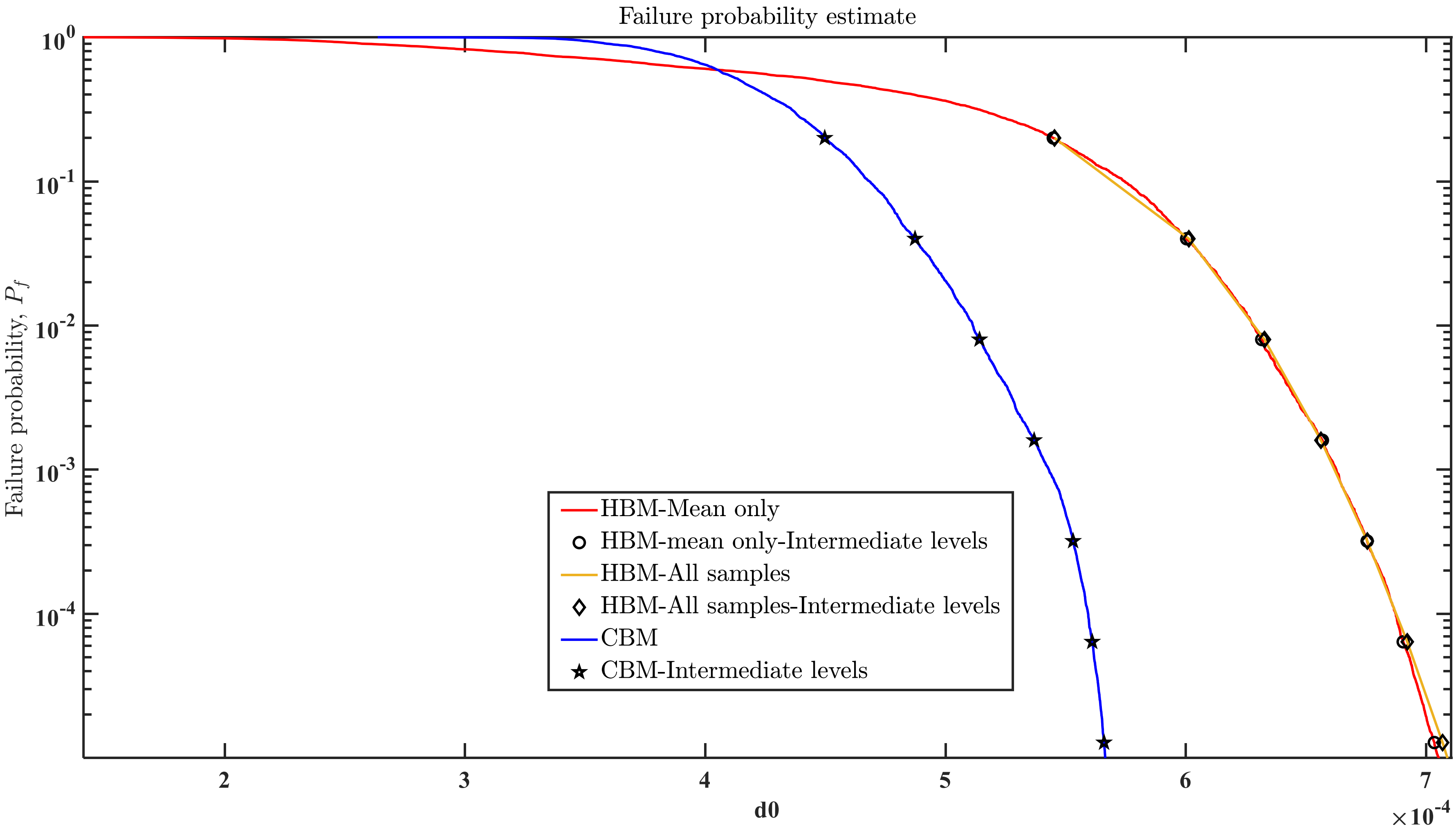

PDF of model parameters for Cases 1-3

# Application to Structural Dynamics




Posterior predictive distribution of the maximum drift for Cases 1-3

# Application to Structural Dynamics




The exceedance probability of the maximum drift for Cases 1-3

# Application to Structural Dynamics

## Experimental example: 3-DOF linear structure with 19 modal properties

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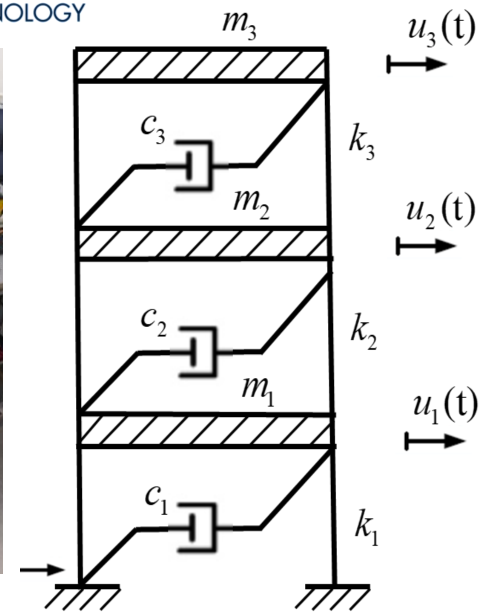
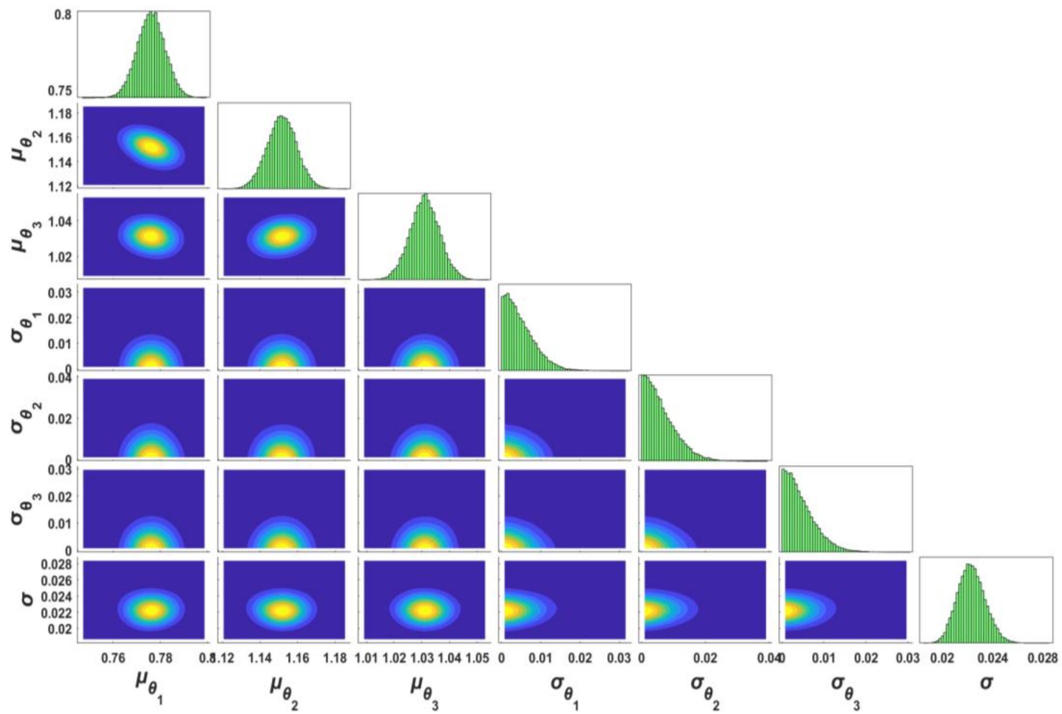
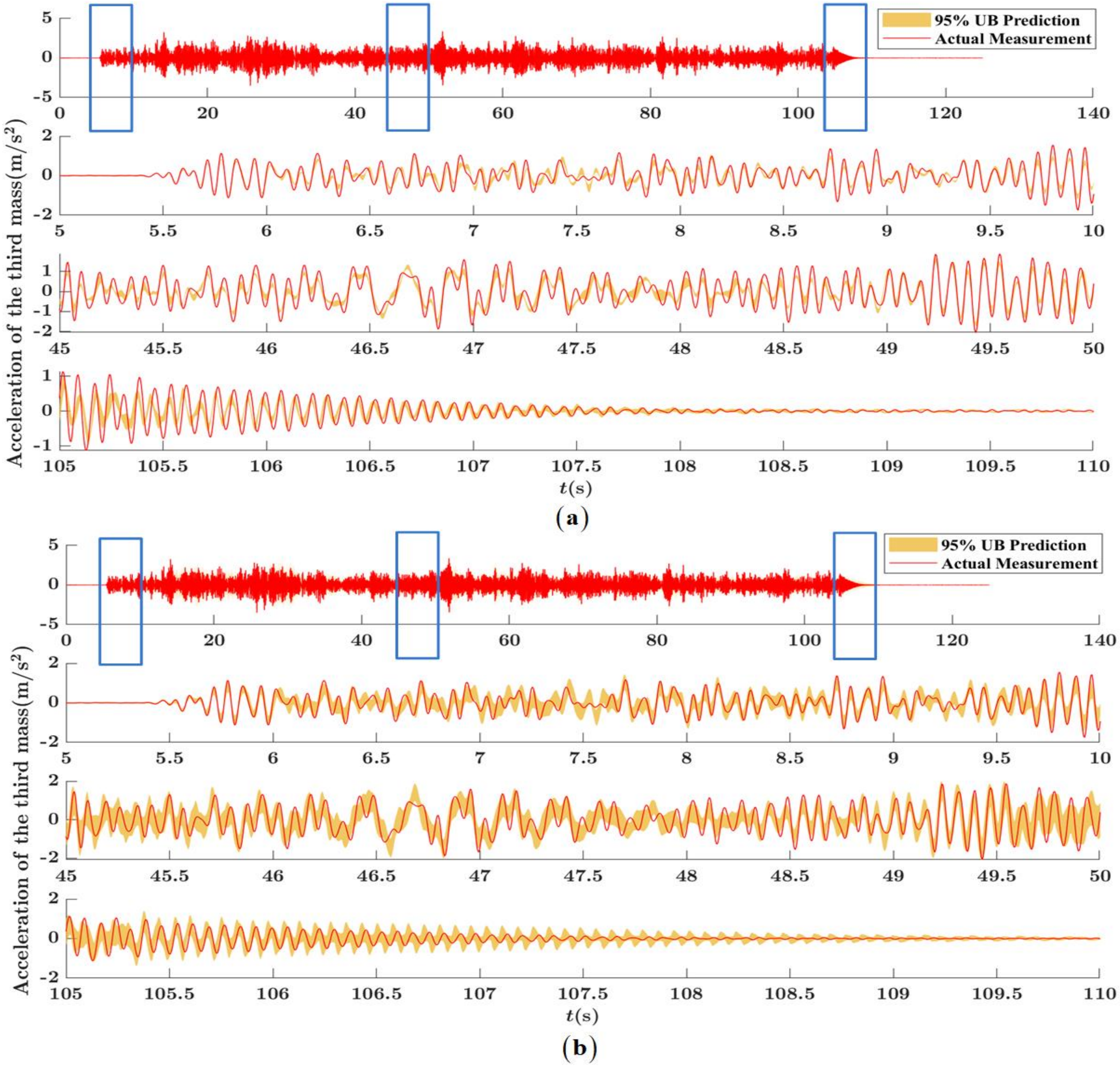



Estimates of mean and standard deviation of the model parameters and prediction error parameters

| | Parameters | $\hat{\mu}_{\theta_1}$ | $\hat{\sigma}_{\theta_1}$ | $\hat{\mu}_{\theta_2}$ | $\hat{\sigma}_{\theta_2}$ | $\hat{\mu}_{\theta_3}$ | $\hat{\sigma}_{\theta_3}$ | $\hat{\sigma}$ | $\hat{\Sigma}_{\sigma}$ |
|---|---|---|---|---|---|---|---|---|---|
| HBM | A-1 | 0.7735 | 0.0068 | 1.1548 | 0.0084 | 1.0301 | 0.0063 | 0.0215 | 0.0010 |
| HBM | A-2 | 0.7757 | 0.0040 | 1.1523 | 0.0041 | 1.0317 | 0.0041 | 0.0223 | 0.0012 |
| HBM | FS | 0.7769 | 0.0043 | 1.1524 | 0.0044 | 1.0309 | 0.0042 | 0.0224 | 0.0012 |
| | CBM | 0.7755 | 0.0032 | 1.1526 | 0.0033 | 1.0317 | 0.0034 | 0.0226 | 0.0012 |

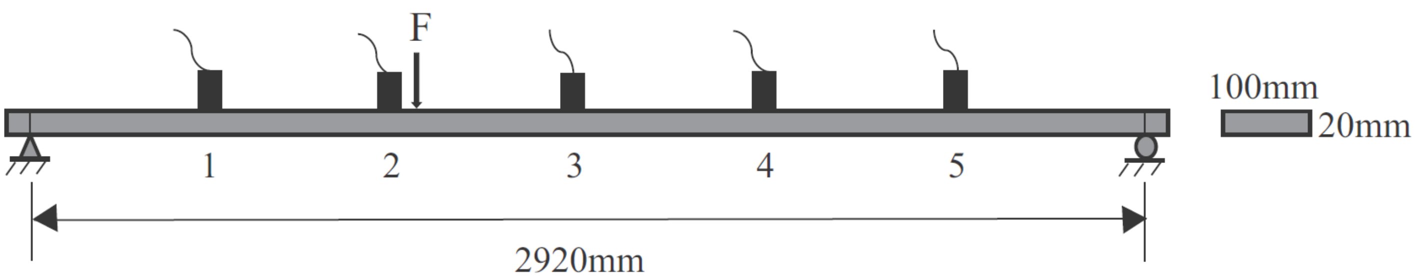
# Application to Structural Dynamics



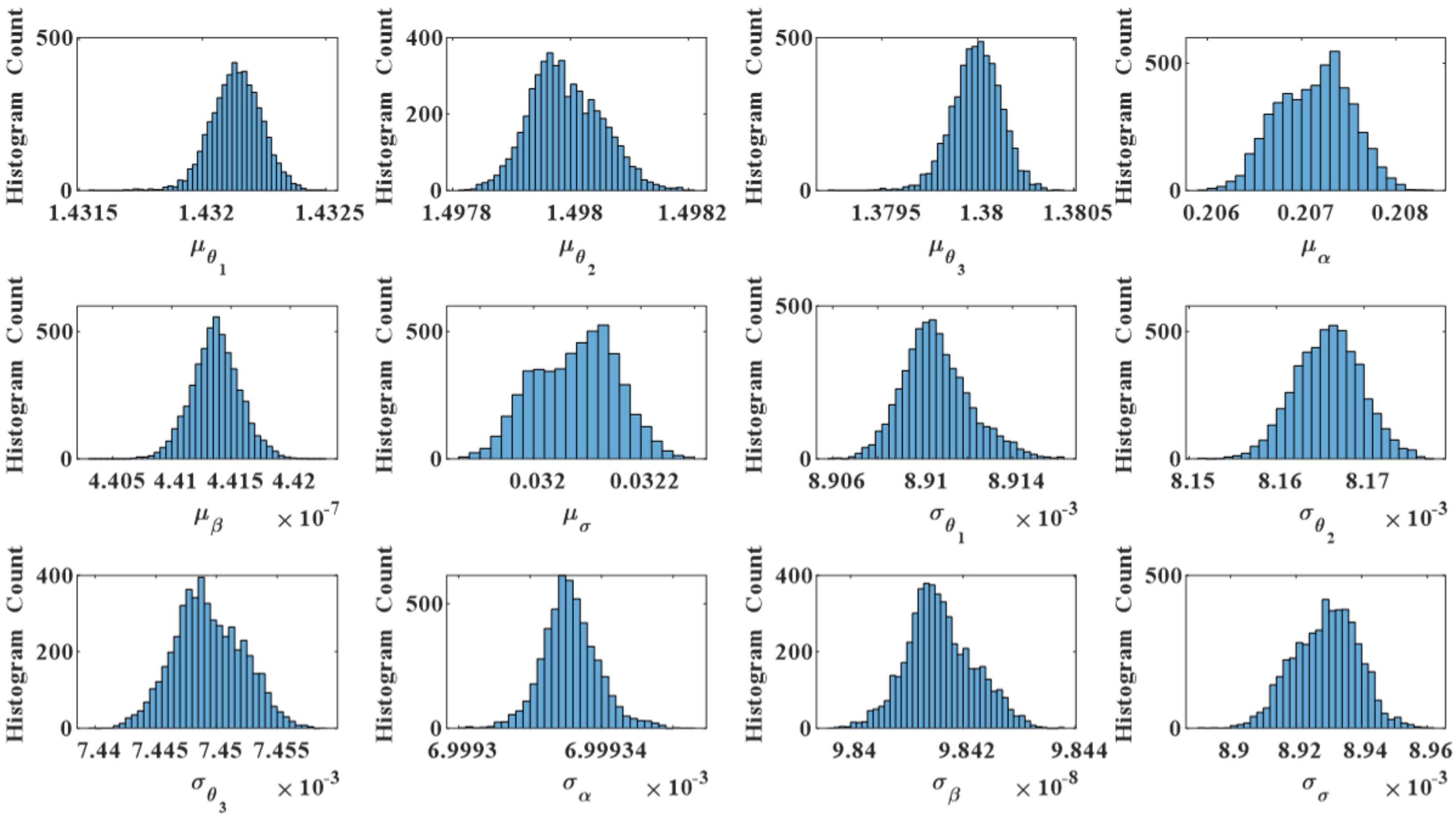

Predicted acceleration response history of the third story considering (a) only structural parameters uncertainty, (b) both structural and prediction error uncertainties

# Application to Structural Dynamics

## Experimental example: Simple supported beam

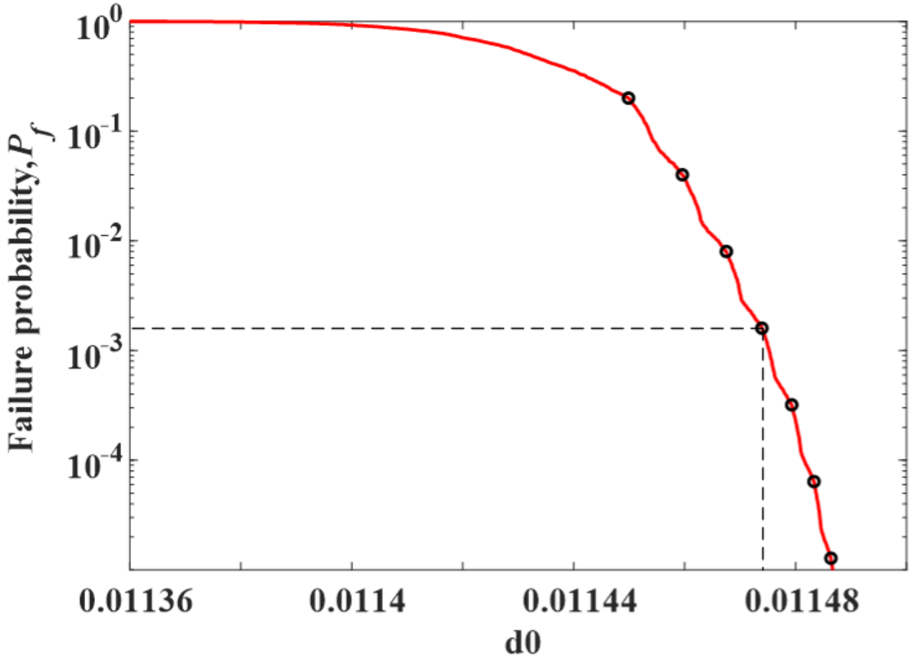



1. Five sensors
2. Nine different frequency response function datasets
3. Five parameters (three parameters related to elastic modules and two parameters in Rayleigh damping model)

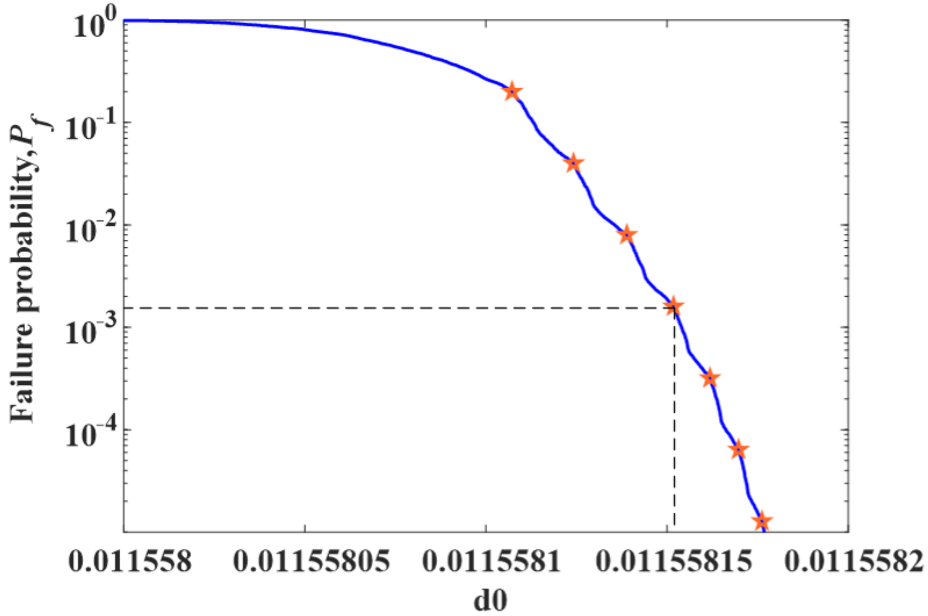



Posterior distribution of hyper parameters using VI

# Application to Structural Dynamics




HBM




CBM

Failure probability of simple supported beam with maximum displacement using (a) HBM (b) CBM

# Conclusions

- **Classical Bayesian framework may severely underestimate uncertainties**
- **Promoted the idea of embedding uncertainties into the system model parameters, introducing a HBM framework**
- **HBM framework provides a more realistic account of the modeling uncertainties. It successfully incorporates uncertainties that are irreducible into the posterior uncertainty**
- **HBM framework is applied to updating reliability based on monitoring data**
- **Framework applicable to general dynamical systems**

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